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Master of Science in Advanced Mathematics and Mathematical Engineering

Title: Museization of Steiner's 10 Questions on the complete quadrilateral

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Supervised by Joan-Carles Lario i Loyo, Anna Rio Doval

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Thanks to my teachers, to my students, to my friends and to my family, without them I wouldn't have had the knowledge, ambition, energy, resilience to do a project like this.

MUSEIZATION OF STEINER'S 10 QUESTIONS ON THE COMPLETE QUADRILATERAL

JOAN ALEMANY FLOS

Abstract. Mathematics are underrepresented in Science and Technology museums. Using Steiner's 10 theorems on the complete quadrilateral as a common trail, the purpose of this paper has been to create an exhibit to illustrate what mathematics are and it's importance and beauty. **Steiner, Complete quadrilateral, Museization, Gauss-Bodenmiller, Newton line**

Contents

Date: June 2019.

1. How to read this paper

The aim of this paper is to create an exhibit around Steiner's 10 Questions on the complete quadrilateral. Steiner was a Swiss mathematician born in 1796 and died in 1863 [1]. In 1827 he proposed in the Annales de Gergonne [2] 10 unsolved questions on this geometrical figure.

This work is distributed in four different sections. Each section is focused on one aspect of the project and is intended to be independent from each other so the reader can decide to read them in the order of their choice.

We don't want to propose an exhibit on Steiner's 10 questions without knowing the current tendencies on science museums and mathematical centers. The **first section** of the paper is centered in how this discipline is treated in Museums around the world. Many places have been visited and analysed during the making of this project, here you can read some of the reflections inspired by them.

We cannot forget the importance of understanding a concept before being able to use it or explain it. The **second section** focuses on the mathematics behind Steiner's 10 questions. We give proofs to questions 1-10 and give a visualization of each one. There has been an effort to search for simple proofs, that do not require hard or long theorems, as it will help create a more approachable exhibit.

The **third section** are the proposals for the exhibits. Two different proposals are given, with different aims and different visions, although both of them have Steiner's 10 questions as a common base. An approximate budget for each proposal is given.

The **fourth** and final section is the creation of a web-app. Based on the concept of Dynamic Geometry, an analysis on the requirements for a program to help visualize the theorems is given and a working demo has been created.

Conclusions, further work and a final though end the paper.

2. The role of Mathematics in museums

There are many methodologies to create an exhibit and to analyse the impact of science museums in society. Wagensberg [3] discusses the difficulty of such analysis, arguing that it is not enough to check the number of visitors and square meters that an exhibit has, many more parameters should be studied. In some cases this data is not even gathered by the museums. Prior to the creation of the proposal of the exhibit for this paper an exploration on the current situation of science museums has been done. The methodology has been similar to the one Riera [4] used to study the museums of Barcelona, but the selection has been decided to be more international. Moreover, there has been a selection of museums depending on the paradigms, Science Museums, Science Centers, and the new mathematical museums [5]. To complete the list of exhibitions, 3 more events have been considered. The complete list is as follows:

- Science Museums / Science Centers:
  - The London Science Museum (England)
  - The Exploratorium in San Francisco (US)
  - Museum of Science and Industry in Chicago (US)
  - Cosmocaixa and MIRRORS exhibit in Barcelona (Catalonia)
- Pure Mathematic Museums:
  - MoMath in New York (US)
  - MMACA in Cornellà (Catalonia)
- Other exhibits:
  - Auto-MATIC at Centre Santa Mònica in Barcelona (Catalonia)
  - Art Institute of Chicago (US)
- Conferences:
  - MATRIX 2018 (Catalonia)

2.1. **The London Science Museum (England).** *Visited February 2018*

This museum located at the heart of London approaches science and technology from the historical side. The collection includes many objects and machines that date more than a century. There are 2 spaces in the museum worth mentioning in particular.

The Winton Gallery [6] is the mathematical gallery of the museum. It includes more than 120 objects (and stories). Their approach is very much applied (there are no "concepts or ideas" but physical objects. When entering the gallery you can read the description:

"People use mathematics in industry, commerce and government, at universities, at home and at play. [...] This mathematical practice has shaped, and been shaped by, some of our most fundamental human concern- money, trade, war, peace, life, death and many others. This gallery presents 21 historical stories about people and their mathematical work over the last 400 years."

The authors of the exhibit only portrait the use of mathematics. The stories presented are mostly descriptive of the historical importance. Some of the stories have an audiovisual where you can see and listen to the explanations. There is almost no interaction with such objects, contained in protected inside a display cabinet. Figure 1 shows the text of the entrance and an example of encased machine that you can look but not touch. The visitors are adults and adults with children, but few visitors stop to listen to the whole video explanation. Few of the objects were linked to geometry.

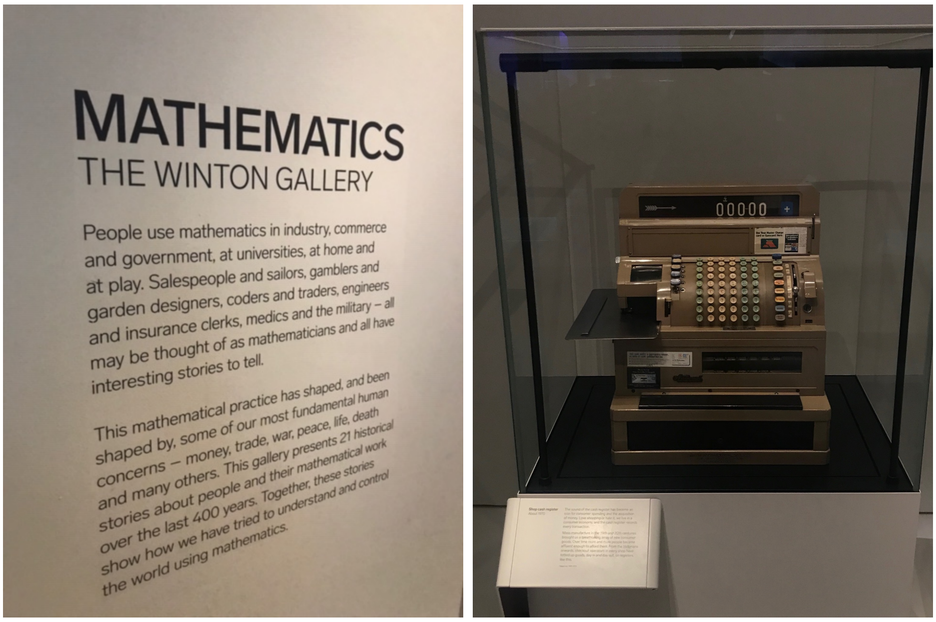


FIGURE 1. Vision of the Winton Gallery and example of object

FIGURE 2. Top floor of the London Science Museum

The most notable ones are some polyhedron, some ruled surfaces, and some drawing made for architecture.

On the top floor of the science museum we can see a completely different exhibition [7]. This exhibition "Engineering the future" is focused on puzzles and games. The interaction is obtained through modern technology. Although the space is recommended for 11-15 year old visitors, we can see a wide range of visitors just playing with the displays. Figure 2 shows some of these displays. The visitors looked to be more engaged than in the Winton Gallery, but one can notice that they do not read the instructions but instead try to play with a trial and error approach.

2.2. **The Exploratorium in San Francisco (US).** [8] *Visited July 2018*

The Exploratorium was created by Frank Oppenheimer in 1969, in San Francisco. As they define themselves in their vision [8], it is a public learning laboratory. In fact this is considered one of the first Science Centers. Compared to Science Museums

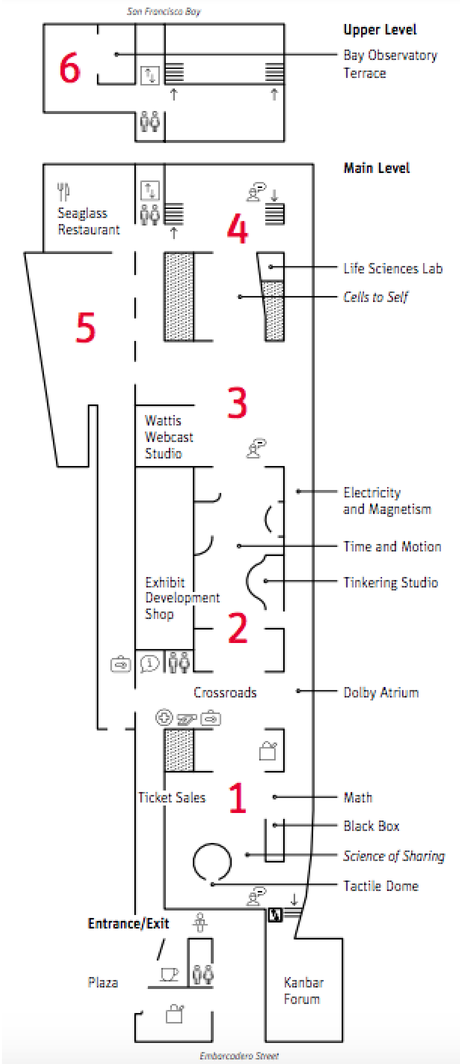


FIGURE 3. Map of the exploratorium

visitor had to take a more active role. They search for an inquiry-based learning. Posing questions to the visitor and giving little answers. In this sense they bring many visitor-centered experiences for them to reflect.

In 2017 the number of visitors was 849,702 and it had 670 exhibits [9]. The average visitors are children. Although the surface of the museum is quite large as you can see in Figure 3, only a small part is dedicated to mathematics. In particular half of area 1. The contents of this part are mainly probability and some number properties on Pi. The only geometry based activity is a display where you can construct some polyhedron from smaller pieces.

Coherently with their vision the exhibits do not give the mathematical answer to the questions they set. They use technology to run some simulations to make the results more impressive. At the same time there is a non automatic experiment based on the Buffon's needle to find an approximation of Pi. These two displays can be seen in Figure 4.

Although their aim is not to give answers, some of the explanations are written to help the visitor understand what they are doing. Not every visitor takes the time to read the explanations as there are other more impressive displays. At the

FIGURE 4. Two displays of the Exploratorium

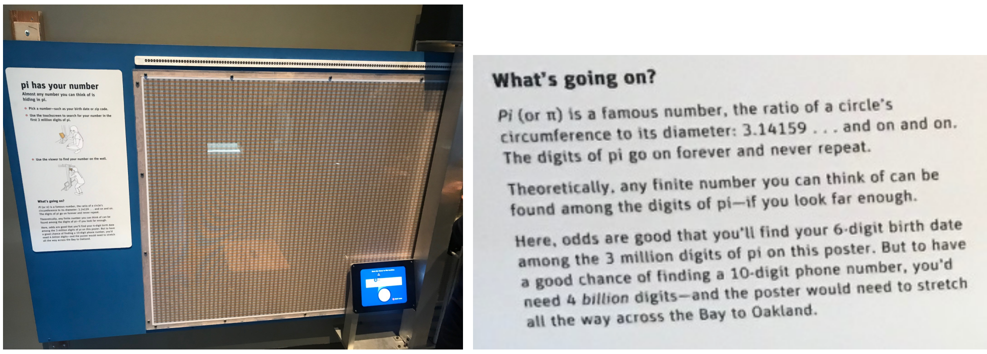


FIGURE 5. A display with an inaccuracy at the Exploratorium

same time we have a concern on some of the explanations they offer. Some of the information shown is not accurate. This is more common in physics display, as there are many assumptions that simplify and approximate the results. In this case it is in a mathematical display related to Pi which can be seen in Figure 5.

It is not proven that "Pi contains any possible string among it's decimal representation". This property for a number is called to be "normal", and in fact it is not proven that Pi is normal. This is still a conjecture, and if it wasn't it should contain the name of the theorem or person that proved it [10].

It is important to notice two more things about the Exploratorium. There are many guides and voluntaries that actively engaged with the visitors. You can see that from the visitors point of view the interaction with them is very positive, as they learn more on the questions posed and they do not switch too fast from activity to activity.

The other thing is that although the space would allow to have bigger displays, the visitors are more engage with smaller individual ones, and the larger ones help to make it more impressive, but less interactive.

2.3. **Museum of Science and Industry in Chicago (US).** [11] *Visited July 2018*

The Museum of Science and Industry was established in Chicago in 1933. Their mission is "to build public understanding of science, show how science impacts society and inspire everyone to think critically about the world around us." This can be seen in their communication approach. The exhibits are very practical and a historical context is given for every display.

FIGURE 6. Displays at MSI Chicago

As they state in their vision, the museum is aimed at inspiring children, giving some of the contexts a childish approach. At the same time an exhibit about the war was a counterexample of this vision. This childish approach to science can be seen in the Mathematics section of the museum. Compared to the other parts it is a rather small area, which includes a mirror labyrinth. One of the consequences is that it is not in an open area and that you need a ticket to enter. There are simple but interesting questions inside the maze, that are not contextualized.

The topic of choice is "Patterns and numbers". Except for the mirror maze, there is a low level of interaction with the visitor, most of it is done with computer displays. These are very guided. The program has very few options and there is no feeling on being able to play with it, just read and click continue.

Something different from the other museums is a wall with pictures where you can identify some of the patterns introduced at the beginning (Figure 6). Although the idea seems nice two major concerns have to be said. First, the wall is only visible if you decide not to enter the labyrinth. Second the explanation is in a midpoint between a kids explanation and an adult one. For example there is a picture of a giraffe with the sentence: "Pigments in giraffe skin and fur follow a Voronoi Pattern". No direct questions are asked (which can help a child engage more directly), and no definition (formal nor informal) is given for Voronoi pattern, which would be nice for an adult audience. Finally because most of the mathematical displays are after the labyrinth the visitor does not pay much attention as they move on to other sections.

2.4. **Cosmocaixa and MIRRORS exhibit in Barcelona (Catalonia).** [12]
Visited January 2019, May 2019

The science Museum in Barcelona was created in 1981 by the "Fundació la Caixa" and it was remodeled from 1998 to 2004. Wagensberg, the director of the museum for 20 years had a clear vision on what science museums had to satisfy: the museum needed to give a bridge between the visitor and the reality through objects. This paradigm is difficult to fit in with the mathematical concept of abstraction. The museum received over 850,000 visitors in 2017, most of which were children [13].

In this museum many displays can be found that cover different aspects of science. Some of the highlights are the Flooded forest, the Geological wall. One of the issues that Riera [4] describes and I agree is the difficulty in separating spaces on the general

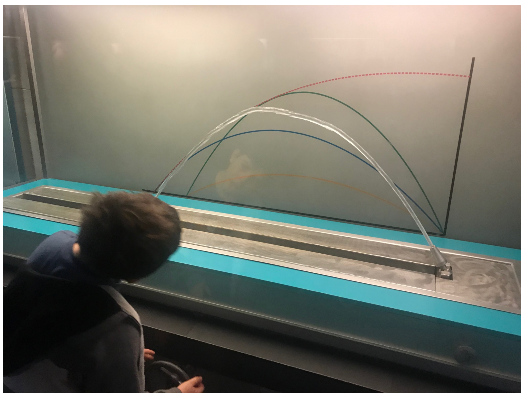


FIGURE 7. Parabolas at Cosmocaixa

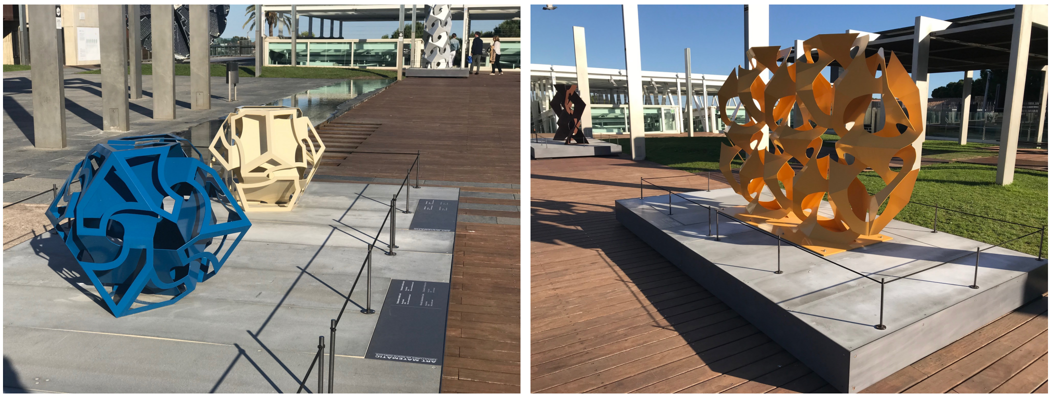


FIGURE 8. Mathematical Art at Cosmocaixa

main floor. This is why there is no particular mathematics section in the permanent exhibit. Nonetheless there are some displays related to mathematics. An automated Galton machine approaches the visitor to the concept of probability, and some rays of water can teach a young visitor about parabolas (Figure 7). It is interesting to state a result also seen in this display (the envolupant of the rays of water is also a parabola) but not explained in detail. This lack of explanation to the visitor makes it difficult for them to engage completely. Although they understand that something is going on, that there is a pattern, it is very difficult for them to keep investigating some of these questions. This is probably intended as Wagensberg wanted the museum to be for everyone, and to be a place where there were more questions than answers.

Some of the temporal exhibits contain some levels of mathematics. For example the MIRRORS exhibit explores some geometric concepts using mirrors. Nevertheless some inaccuracy may be found in the museum, similar to the case of the Exploratorium some of the explanation are not true although they are widely accepted in physics. This is important if we want to empower the rigour of science, but it simplifies reality so that it can be better understood.

Another temporal exhibit worth mentioning is Mathematical Art, where the museum exhibits some sculptures by the artist and mathematician Rinus Roelofs. Although there are few explanations around the creation of the objects, they are inspiring and give an open approach to mathematics. Figure 8 shows some of these sculptures.

FIGURE 9. Large displays at MOMATH

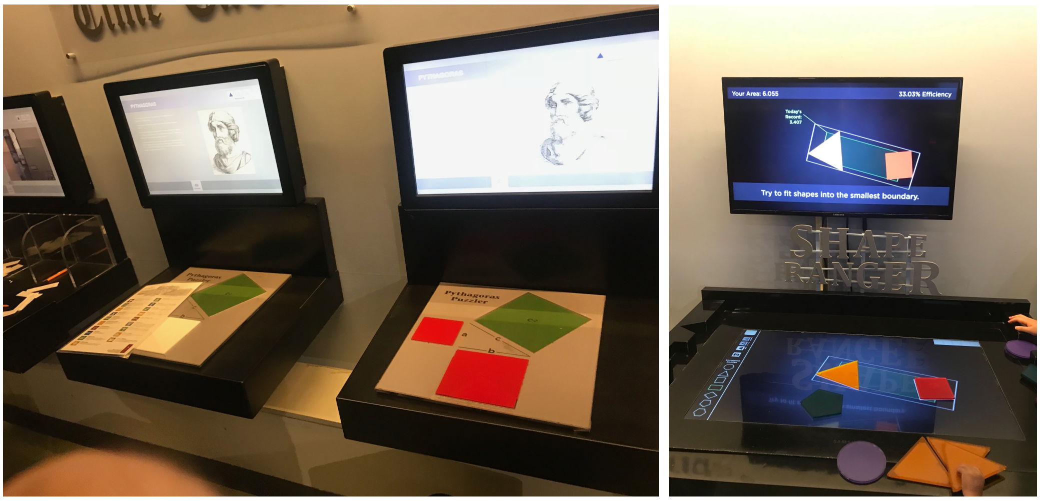


FIGURE 10. Individual displays at MOMATH

It is important to mention that the museum also celebrates regular conferences, as well as other activities. In Wagensberg vocabulary, it is an important center for the "spoken" transmission of knowledge.

2.5. **MoMath in New York (US).** [14] *Visited July 2018*
The National Museum of Mathematics (MoMath) is located in Madison Square Park, at the center of New York (USA). It is a modern museum, that combines a permanent exhibit for children with conferences and activities for adults. They have a broader vision than other museums (which only focus on children). The reason is that they wanted to broaden the vision that Goudreau Museum in long island had. As they explain in their history, the idea of MoMath comes after the closure of this mathematics museum, as they left an empty space of knowledge, and sharing to cover [15].

One of the first characteristics one can notice when entering this two stories museum is the lack of space, compared to any of the previous museums that were build on open wide spaces. This is compensated by a good design of displays that cover all the area. There are displays of multiple sizes. Some of them are very large (Figure 9), but there is also space for smaller individual displays (Figure 10).

The majority of displays use technology, but do not forget about other manipulative materials. They are hands-on activities. There is a variety of concepts treated around

FIGURE 11. Floor activities at MOMATH (Right) and MMACA (left)

the museum, but there is a lack of explanation on the mathematics behind each display. This again is coherent with their vision, inspire questions to the visitor, but do not give the answers.

On the lower floor we can see many different sections. Two of them are worth mentioning because of the engagement you can see from the visitors. These are the "Enigma Cafe" and the "Math Square".

The "Enigma Cafe" is a non-technology space located in the center of the bottom floor with tables and wooden puzzle games distributed around. The visitors seat down and try to solve the challenges proposed. The attitude is more engaged than in the other displays, and they seem to take longer to move to a new section.

The second section, the "Math Square" is an interactive floor platform that can detect and change according to the visitors. There are activities related to Voronoi, and some puzzles. The visitors (mainly children, but also adults) walk around playing with the lighted floor. Figure 11 shows an example of puzzle.

2.6. **MMACA in Cornellà (Catalonia).** [16] *Visited May 2018, March 2019*
The Museum of Mathematics of Catalunya (MMACA) is located in Cornellà Catalonia. The museum started in 2008, but it wasn't until 2013 that the city of Cornella gave them a space in the Palau Mercader, a historical site inside a public park. This is a very interesting location because in one hand it limits the space of the museum, but at the same time because it is inside a park, it allows the museum to explore activities on the outside. Figure 12 shows Leonard Domes, one of the activities done outside. The museum is also in charge of doing conferences for adults.

The mission of the museum is to foster the learning of mathematics, but their vision is slightly different to other museums. There are no fancy technological displays, most of them are for individual use, portable (the museum started as an itinerary exhibit), and made of wood. Two examples of displays can be seen in Figure 13. In fact there is only one display which is electrical, and it is a collaboration display with the MoMath (originally called the Ring of Fire).

Finally they have some large floor platforms that have games for the kids to explore physically. This can be seen in Figure 11.

Similarly to what Wagensberg said [3] in his reflections of the complete museum, almost all the displays do not require any knowledge to play with them and pose questions or challenges to the visitor. At the same time, very few of them treat abstract thinking. For this reason, and to promote mathematics in social networks a weekly problem publication has been created with the name of EnigMMACA.

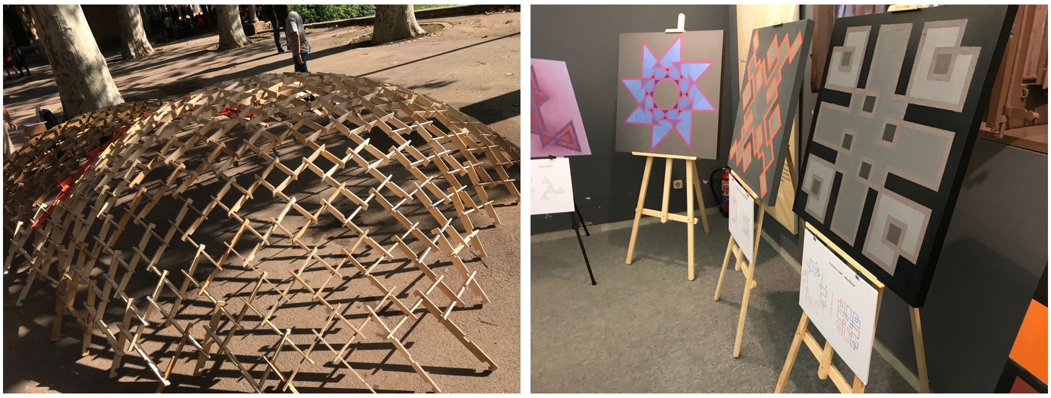


FIGURE 12. Leonardo Domes and MMACA conferences

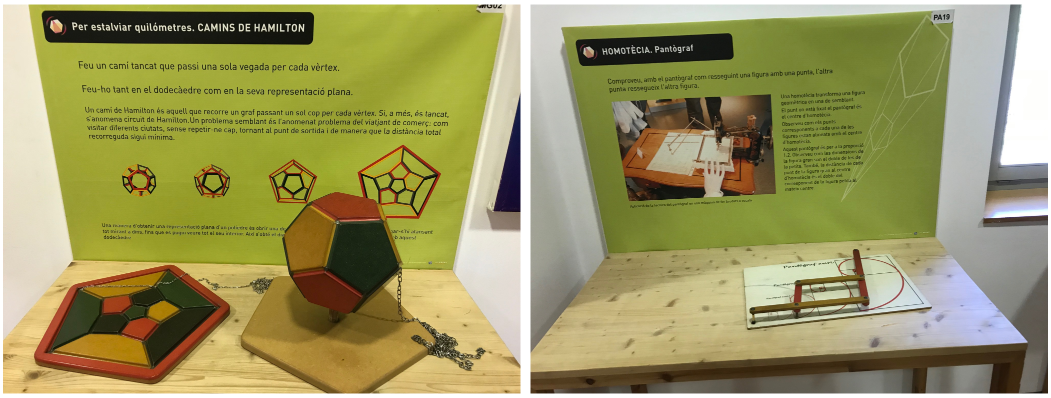


FIGURE 13. Displays at MMACA

FIGURE 14. ENIGMMACA on twitter. Geometric problems for all.

Figure 14 shows some of these challenges. Because of the visual and simple representation of geometry, most of the problems have covered this topic.

2.7. **Art Institute of Chicago (US).** [17] *Visited July 2018*

This is a non-science, non-mathematics museum. As they write in their mission: "The Art Institute of Chicago collects, preserves, and interprets works of art of the highest quality, representing the world's diverse artistic traditions, for the inspiration and education of the public and in accordance with our profession's highest ethical standards and practices." It is important here to mark the "inspiration and education" of the public. This is one of the main differences between art museums and science museums, as it doesn't try to set up questions but to inspire. The museum was build

FIGURE 15. Lázló Moholy-Nagy and Vincent VanGogh Paintings

in 1879 and hosts more than 1.6 million guests annually [18], although we can read in the annual report that each visitor takes in average less than 30 seconds per painting.

The space of the exhibit is wide, similar to the science museums visited. You can see that the art pieces are the center of each room. Very few text covers the walls, no explanations are required, some minor historical contexts are explained. Very few comments are made on the techniques used to achieve these pieces of art. The behaviour of the visitor is more contemplative than in science museums. The average public are adults. The guided tours (or audio-guide) complement the context, techniques, and other information to the interested visitor.

On the content of the painting, we can see that some of them contain geometrical shapes, either because of the abstraction as the A19 painting by László Moholy-Nagy or because they use perspective (where the complete quadrilateral would appear). We know that Vincent VanGogh explored the mathematics and the techniques involved in perspective for his paintings [19]. These examples can be seen in Figure 15.

2.8. **Auto-MATIC at Centre Santa Mònica in Barcelona (Catalonia).** [20] *Visited October 2018*

This temporal exhibit held in Centre d'Art Santa Mònica in Barcelona, consist of 120 drawings created by computers, humans and machines. The project director was Edouard Cabay, and the exhibit was created during 3 years. The exhibit had a modest set up (Figure 16), and covered the second floor of the venue. The visitors were mainly adults.

The exhibit could be considered both an art exhibit and a mathematical (computer science) one. The displays are non-interactive (similar to the art exhibits) but there is a "scientific" explanation on the creation of the piece. This type of art is considered to be generative art, which is art created with the help of a computer program or robot. The creation process for these drawings was more important than the final outcome of it [21]. This approach can be interesting for a mathematical exhibit, where the process to reach an outcome is as important as the outcome itself.

Although there is very few explanations on the mathematics behind the art pieces, there are explanatory videos next to each "experiment" and a rigorous description of the algorithms used. In Figure 17 an example of Algorithm and Machines can be seen.

FIGURE 16. AutmoMatic Exhibit

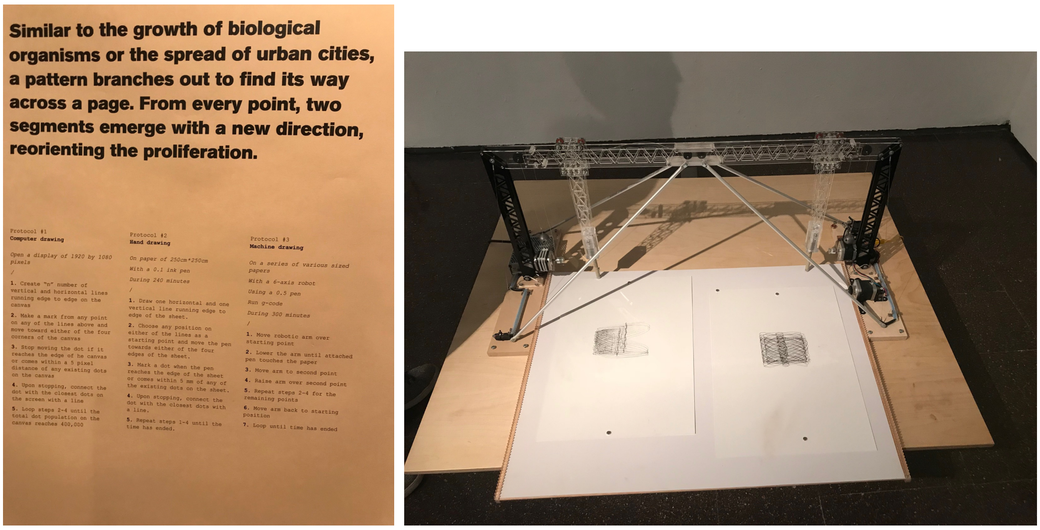


FIGURE 17. An algorithm for Computer, Hand and Machine at the AutoMatic Exhibit

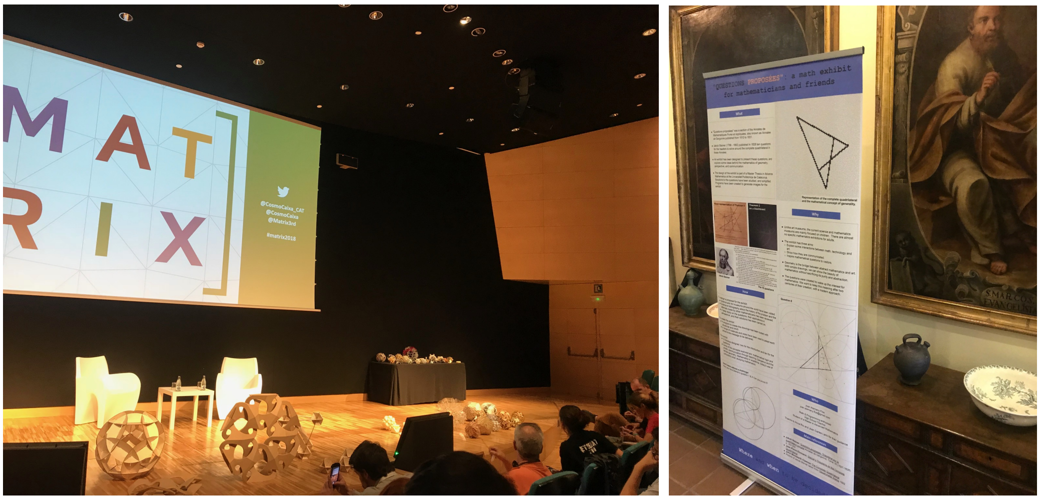


Figure 18. MATRIX 2018 Conference and poster Fair

2.9. MATRIX 2018 (Catalonia). *Assisted October 2018*

MoMath and MMACA organized in October of 2018 the III International Conference MATRIX, aimed at mathematics museums and outreach centers. The opportunity to attend this meeting was invaluable for this paper. I presented a first proposal for the exhibit on the poster Fair to obtain some comments on the work done so far (Figure 18).

Although there was no specific exhibits, there were many ideas and philosophy to take home. The conference of Arts and Maths by Rinus Roelofs was inspiring, on how maths could help on the creation of art. This was already a topic that I decided to include as Van Gogh was learning perspective to create his paintings.

The work groups were interesting points to gather other points of view. One of the topics which was important was from "Is virtual virtuous?" as it is a controversial topic in science museums (that want to avoid the abstraction of virtualization). Also about the creation rights of exhibits. In these sense I learned about the declaration of Dresden which I think it will be something to include in the rights of usage of the exhibit [22].

3. Mathematical approach to Steiner's 10 Questions

As we have seen in the first section, abstract thinking and proofs are underrepresented in science museums. Geometry is a good topic to treat both. On one hand it has a historical importance in mathematics. On the other hand, some of the proofs do not require intricate theorems, or a lot of previous knowledge to understand. There is also visual images that can help understand the theorems (which does not happen with other disciplines of mathematics).

Although one could chose a topic like *Euclid Elements* one of the most important and influential books in history, it would be too broad, and maybe already too known to catch the interest of the visitor. For this reason a different topic is explored: Steiner's 10 theorems on the complete quadrilateral.

Steiner's 10 questions were presented at the *Annales de mathématiques pures et appliquées* in 1827 [2] on a section named Questions proposées (Proposed questions). The aim of this part of the Annales was to set up problems for the readers to prove. Although in general this section was anonymous, at that moment J. Steiner was already an important geometer and the editors of the publication decided to give him credit.

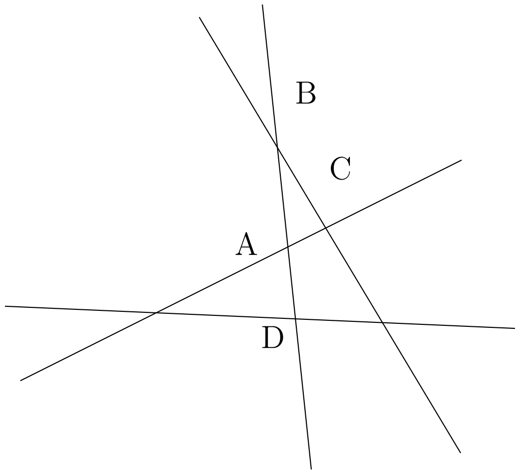
The following is a translation of the section:

Theorems on the complete quadrilateral

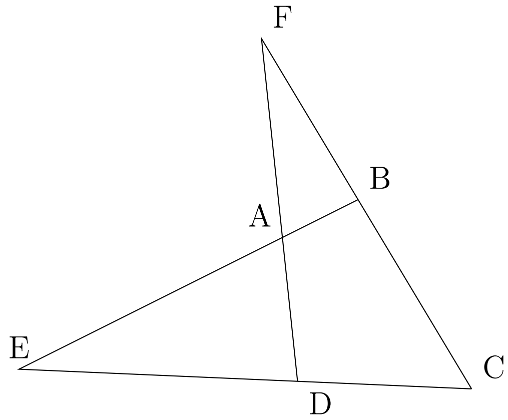
Four lines **A**, **B**, **C**, **D**, intersecting two by two in six points and, in consequence belong to a same plane.

(1) These four lines, taken three by three, form four triangles whose circumscribed circles meet at a common point **P**.
(2) The centers α, β, γ, δ with point **P** lie on a fifth circle.
(3) The feet of the perpendiculars to the directions **A**, **B**, **C**, **D** from **P** belong all four to the same line **R**, this property is exclusive for point **P**.
(4) The meeting points of the perpendiculars from the vertices to the opposing sides of the four triangles (1) belong to the same line **R'**.
(5) Lines **R** and **R'** are parallel, and line **R** goes through the middle of the perpendicular from **P** to **R'**.
(6) The midpoint of the diagonals of the complete quadrilateral created by the four lines **A**, **B**, **C**, **D**, belong all 3 to the same line **R"** (*Newton line*).
(7) Line **R"** is a common perpendicular to both lines **R** and **R'**.
(8) For each of the four triangles (1) there is an inscribed circle and three excircles, which makes in total *sixteen* circles; the centers of which are four by four in the same circle, creating *eight* new circles.
(9) These new eight circles can be divided in two groups such that each of the four circles in one these group intersects orthogonally all the circles of the other group; we can conclude that the centers of the circles of both groups belong to two lines one perpendicular to the other.
(10) Finally these last two lines meet at point **P**, mentioned previously.

These questions have been proved in the past among others by Mention [23], Marchand [24], and compiled by Coolidge [25] or by Wentworth [26]. It is an aim of this paper to give basic proofs of the 10 theorems, and show a visualization of each step. It is for this reason that for each question we will give a figure illustrating the visual representation of the theorem, and we will give a brief explanation of the previous knowledge required to prove it.



(A) Steiner's line notation



(B) Point notation used in this paper

Figure 19. Complete Quadrilaterals

Steiner starts the questions by defining the main figure where he will construct the theorems. *"Four lines A, B, C, D, intersect two by two, in six points and consequently lie in the same plane"*. This is the definition of complete quadrilateral.

Figure 19A shows a representation of a complete quadrilateral using Steiner's notation. To simplify figures and proofs we will use point notation (instead of naming the lines), and we will draw only the segments unless otherwise required. This notation can be seen in 19B.

Definition 3.1 (Complete Quadrilateral)**.** A *complete quadrilateral* is a set of 4 lines that intersect in 6 different points when taken in pairs.

It is important to notice that these 4 segments generate 4 triangles, with 2 different types of behaviour. The smaller triangles ABF and ADE, and the two large ones FDC and ECB. This *relationship* allows us to prove things for FDC and ABF, and assume that the arguments are similar for ECB and ADE.

We will assume that the lines are in general position, that is that there are no parallel lines, and thus do not intersect in 6 points (in the Euclidean plane). Some particular cases could be studied more in depth. For example:

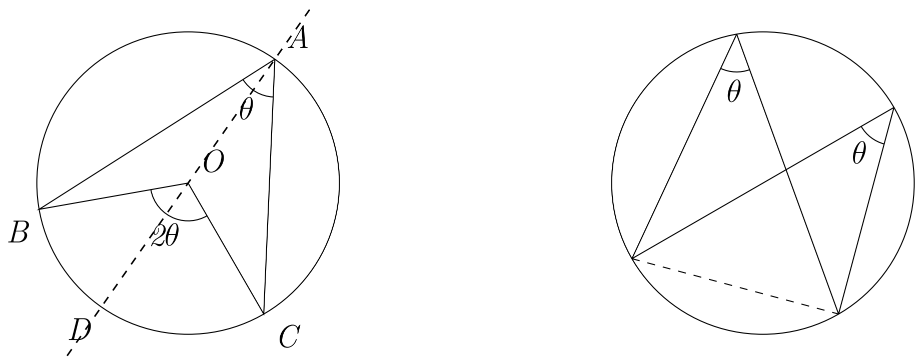
- Fixed lengths: Triangle CEF could be equilateral, ABCD could be a rombus, etc.
- Fixed angles: Angle DCB could be right, line FD $\perp$ EB, etc.
- Relationships with circles: Cyclic quadrilateral ABCD, ABCD inscriptible quadrilateral, etc.

These are sub-cases of the general case, and thus all the properties of the 10 questions are true for them as well. In fact, because it is an additional restriction on our problem we could give shorter proofs of some of the questions or investigate some properties more in depth.

3.1. **On angles and circles. Giving answer to Questions 1 and 2.**

From Steiner's 10 questions, the firsts are provable with very few geometric concepts and arguments. For this reason a step by step proof will be given, which could be included in one of the museum proposals. We should start with a couple of Lemmas.

Lemma 3.2 (Inscribed angle)**.** *The inscribed angle is double of the central angle.*



(A) Inscribed and central angles relationship (B) Angles that cover the same arc

FIGURE 20. Inscribed angle

Proof. Consider Figure 20A Triangle BOA is isosceles, and thus $\angle BAO = \angle OBA$ (the same for triangle AOC). The sum of the angles of ABO is $\angle OBA + \angle AOB + \angle BAO = \pi$, which is $2\angle BAO + \pi - \angle DOB = \pi$ and thus, $\angle BOD = 2\angle BAD$, and so, $\angle BOC = 2\angle BAC$.

□

Corollary 3.3. *Inscribed angles that bound the same arc have the same measure.*

Corollary 3.4. *If 2 triangles with a common side have the same opposite angle, then the four points lay on a circle (Figure 20B).*

Corollary 3.5. *If the central angle is flat then the inscribed angle is right.*

Lemma 3.6 (Points in the same circle)**.** *A quadrilateral ABCD is cyclic (points A, B, C and D belong to the same circle) if and only if opposite angles add up to π.*

Proof. Following Figure 21, the arcs covered by the central angles ∠BOD and ∠DOB add up 2π. Each part being double than the inscribed angle, the sum of the inscribed angles will be π. □

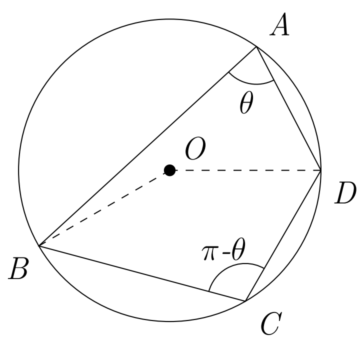


FIGURE 21. Cyclic quadrilateral

Steiner's Question 1. The four lines of a quadrilateral form four triangles whose circumscribed circles meet at a common point **P** (see Figure 22).

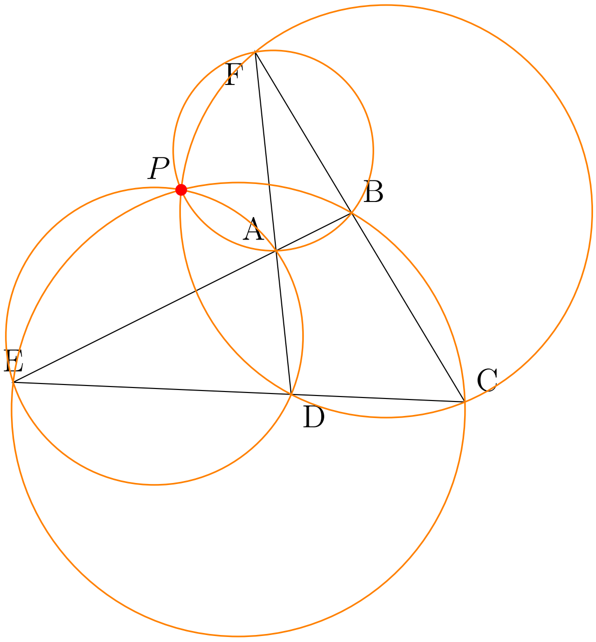
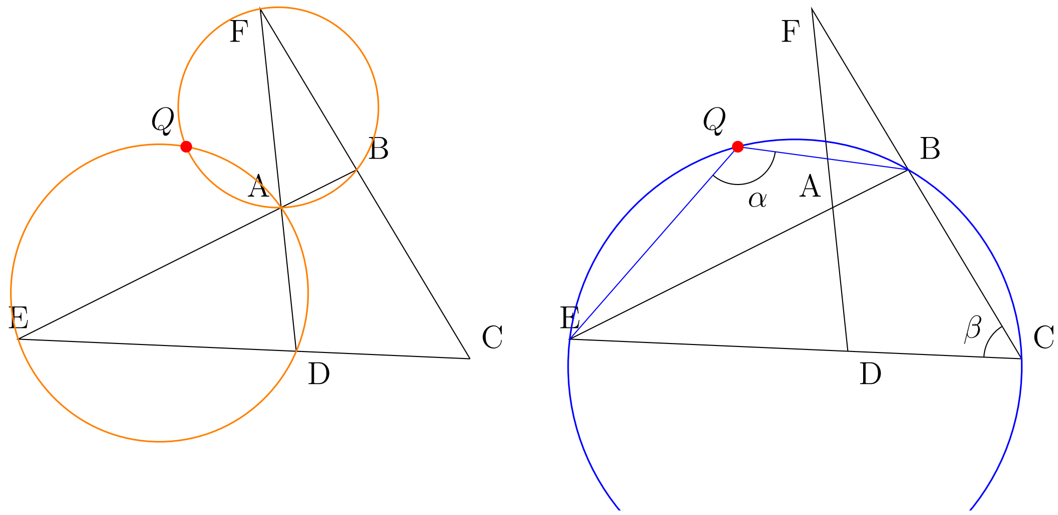


FIGURE 22. Steiner's Question 1

This point **P** is called the focal point [26] or the Miquel point of the complete quadrilateral, as it was proven by A. Miquel in 1838 [27].

Proof. First of all let us consider only the circumcircles of triangles AED and FBA whose intersection (a part from A) we name Q (Figure 23A). We want to prove that Q also lies in the circumcircle EBC (Figure 23B), that is $\angle\text{EQB} + \angle\text{BCE} = \pi$.



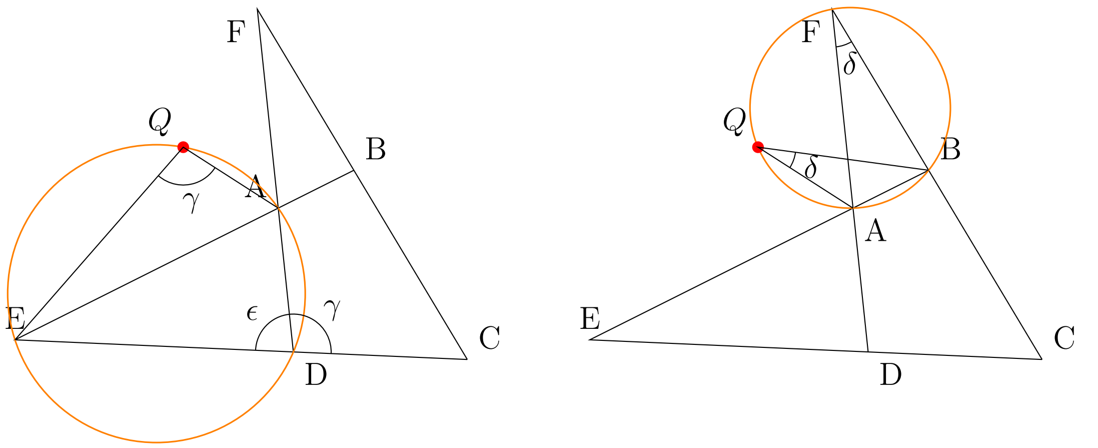
(A) Defining Q from 2 of the circumcircles (B) Does Q belong to the blue circle?

FIGURE 23. Setup and goal of the proof

The point Q belongs to the circumcircle of triangle EAD that means that EQAD is a cyclic quadrilateral, so $\angle\text{EQA}$ is supplementary from $\angle\text{ADE}$. We can see that

∠CDF is also supplementary to ∠EQA, that means ∠EQA is the same as ∠CDF (Figure 24A).

Let us focus on the other circle, we can see both ∠AQB and ∠AFD share the same chord from the same side, this means by Corollary 2 of Lemma 3.2 that they measure (see Figure 24B).



(A) $\angle EQA = \angle CDA$ (B) $\angle AQB = \angle AFB$

Figure 24. Equal angles

The sum of the angles in a triangle is π and thus, focusing on triangle DFC we can see that $\angle FCD$ is $\pi - \gamma - \delta$. We can see in Figure 25 that angle ∠EQB is suplementary BCE which means by Lemma 3.6 that Q belongs to circumcircle of triangle EBC. A similar argument can be used for circumcircle of FDC. Renaming Q to P solves Steiner's Question 1.

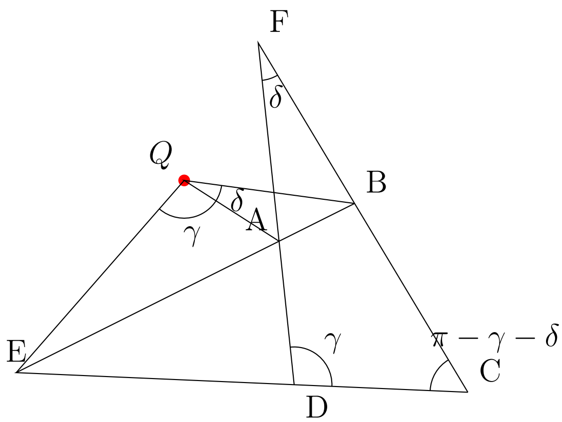


Figure 25. Angle EQB is supplementary to angle BCD

□

Steiner's Question 2. The centers of the circles in Steiner's Question 1, α, β, γ, δ with point **P** lie on a fifth circle (see Figure 26).

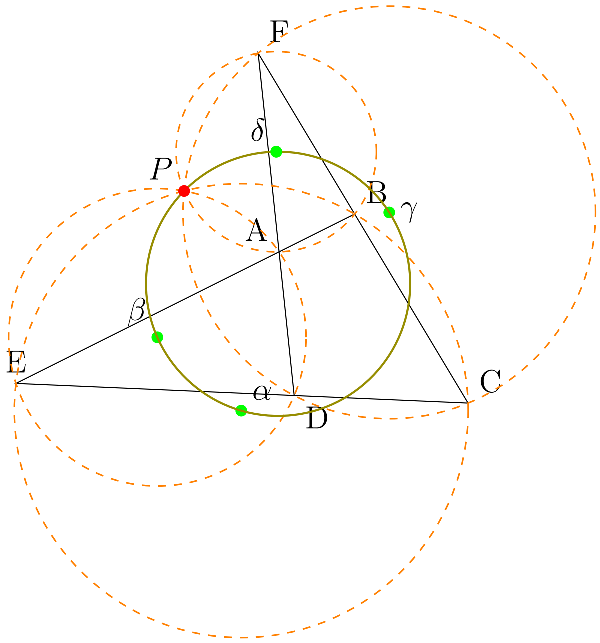


FIGURE 26. Steiner's Question 2

Proof. We will prove that the angles $\angle P\beta\delta$ and $\angle P\alpha\delta$ have the same measure and thus by Corollary 3.4, as they cover the same arc, we will conclude that the four points belong to the same circle.

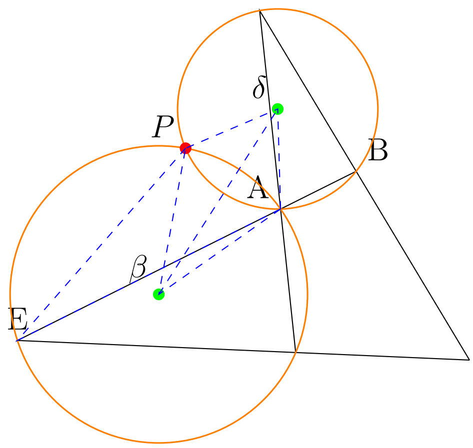


FIGURE 27. $\angle$ AEP = $\angle$ Pβ δ

We have that in Figure 27 $\angle AEP = \frac{1}{2}\angle A\beta P = \angle\delta\beta P$. The first is due to the inscribed/ central angle Lemma 3.2, and the second due to triangle $P\beta\delta$ being symmetric to triangle $A\beta\delta$.

Similarly we can show that $\angle$ BEP $= \frac{1}{2}$ $\angle$ EβP $=$ $\angle$ Pβ δ. The lines to help visualize this are displayed in Figure 28.

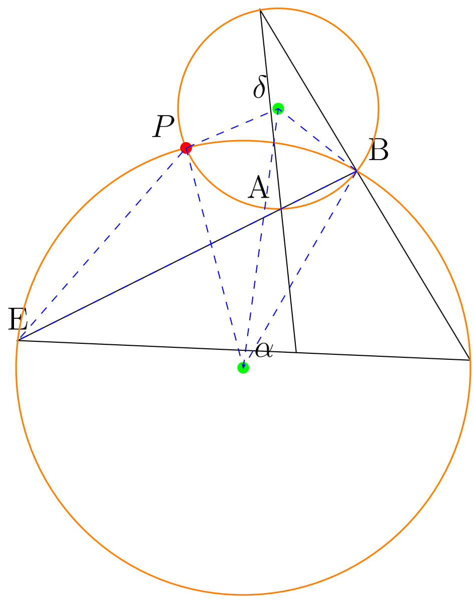


FIGURE 28. ∠ BEP = ∠ P α δ

Because A, B and E are aligned ∠ AEP = ∠ BEP and thus ∠ Pβ δ = ∠ Pα δ. Which means by Corollary 3.4 P, β, α δ are in the same circle. A similar argument can be used to conclude that γ belongs to the same circle.

□

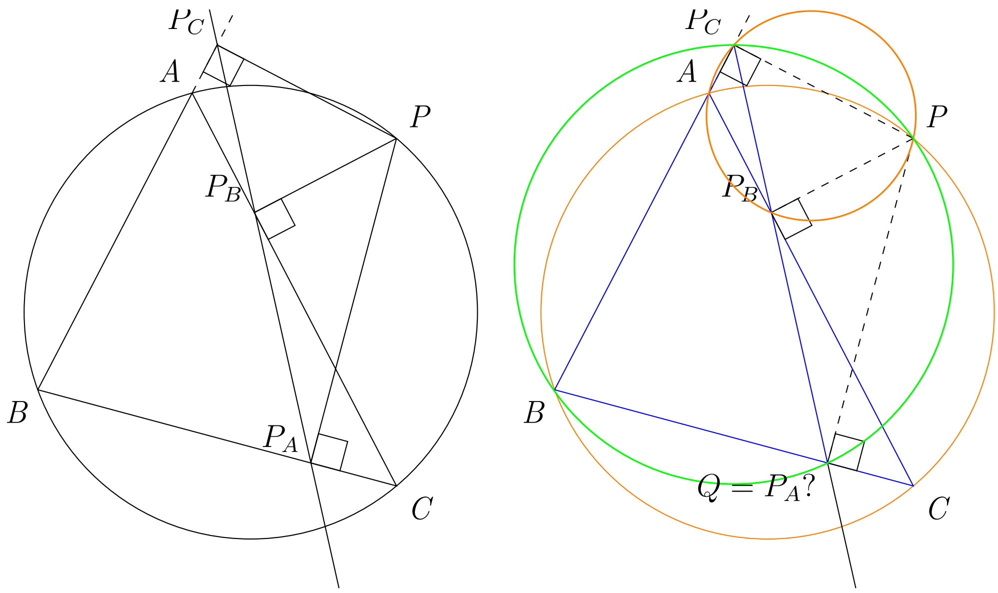
3.2. **On perpendiculars to the sides of triangles. Giving answers to Questions 3, 4 and 5.**

Many notable results can be stated that relate the sides of a triangle and their perpendiculars. In particular we will describe two. The first result is named after Robert Simson, a Scottish mathematician (1687-1768), and it has to do with perpendiculars from a point in the circumcircle to the sides of a triangle [28].

Lemma 3.7 (Simson's Line)**.** *Given a triangle ABC and a point P on its circumcircle, the feet of the perpendiculars from P to the sides (P_A, P_B and P_C) are aligned (Figure 29A).*

Proof. We can use Steiner's Question 1 to prove Simson's Line Lemma. Without loss of generality, denote by P a point in the arc AC, and build the perpendiculars to sides BC, AC and AB. Let P_A, P_B and P_C be the feet of these perpendiculars, and let Q the point of intersection of BC and lines P_B P_C.We will prove that Q is P_A.

Using Figure 29B as a reference, consider the complete quadrilateral with vertices A, B, Q, C, P_B and P_C (all in blue). We can see that the quadrilaterals AP_BPP_C is inscriptible because the angles $\angle AP_CP$ and $\angle AP_BP$ are right, and thus add to π. By Lemma 3.6 P belongs to two circumcircles (orange circles) of the complete quadrilateral, and it is not one of the vertices. That means that it is equivalent to point P on Steiner's Question 1. That implies P belongs to the other two circumcircles; in particular of AQP_C (green circle). Using quadrilateral P_CBQP, we can



(A) The Simson line with respect to P (B) Lines and circles for the proof

see that $\angle PQB + \angle PP_CB = \pi$, and since the perpendicularity of PP_C and P_CB, it follows that $\angle PQB + \frac{\pi}{2} = \pi$ and so $\angle PQB = \frac{\pi}{2}$. This means that Q is the foot of the perpendicular. So $Q = P_A$. That means P_A, P_B and P_C are aligned. $\square$

Corollary 3.8. *The converse is also true. If from a point P we draw perpendiculars to the sides and the feet of these perpendiculars are aligned then the point P is in the circumcircle.*

We shall need another result related with the orthocenter of the triangle, that is the intersection of the heights. There is a direct relation between the orthocenter and Simson's line.

Lemma 3.9. *Given a triangle ABC, its orthocenter H and a point P on the circumcircle, the Simson's Line from P to ABC bisects the segment from PH.*

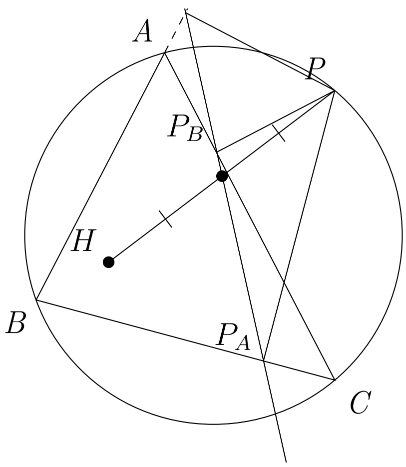


FIGURE 30. The Simson line bisects the segment HP

The proof can be read in Honsberger [29], *in Chen's* [30] *or in Coxeter* [31].

Steiner's Question 3. The base of the perpendiculars to the lines **A**, **B**, **C**, **D** from the focal point **P** belong all four to the same line **R**. Point **P** is the only point with such a property (see Figure 31). The line **R** is called the *pedal line*.

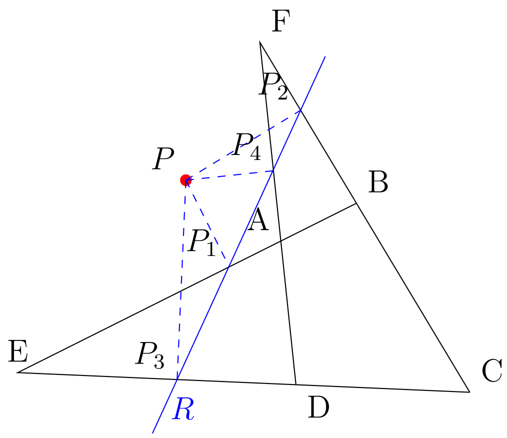


FIGURE 31. Steiner's Question 3

Proof. Recall from Steiner's Question 1, the focal point P belongs to the four circumcircles. Let P_1, P_2, P_3 and P_4 be the projections of P to the lines AB, BC, CD and AD respectively. Consider the triangle ABF, by Lemma 3.7 Points P_1, P_2, and P_4 are aligned, as they belong to the Simson's line from P.

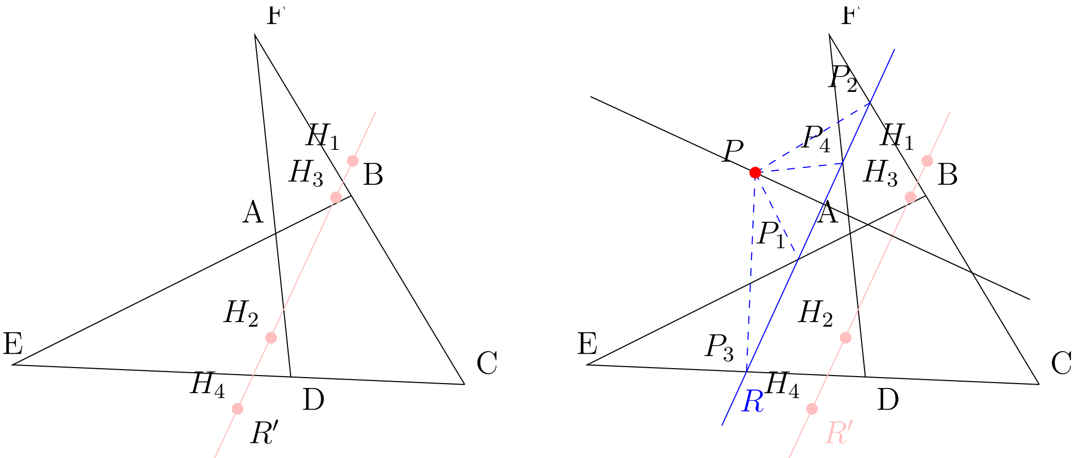
Consider now triangle AED, by Lemma 3.7 the points P_1, P_3, and P_4 are also aligned. This means that the four points have to be all aligned.

Uniqueness: For the projections from a point P' to the sides of a triangle to be aligned, the point must be in the circumcircle due to the corollary of Lemma 3.7. The uniqueness of P comes from fact that it is the only point belonging to the four circumcircles. $\square$

Steiner's Questions 4 and 5 will be consequences of Lemma 3.9 and thus we will prove both at the same time. Let us recall both statements and proceed to the proof.

Steiner's Question 4. The orthocenters of the four triangles (1) belong to the same line **R'**. Line **R'** is called the *orthocentric line* of the complete quadrilateral (see 32A).

Steiner's Question 5. The pedal line **R** and the orthocentric line **R'** are parallel, and line **R** goes through the middle of the perpendicular from **P** to **R'**. Figure 32B



(A) Steiner's Question 4 (B) Steiner's Question 5

FIGURE 32

Proof. Suppose that the orthocenters are not aligned. Consider the midpoints from P to each orthocenter. These points are not aligned, but by Lemma 3.9 they all lie in the Simson's line of each triangle with respect to P. From Steiner's Question 3 we know that the 4 Simson lines to the triangles are the same. This means that the midpoints and hence the orthocenters have to be aligned. This is a contradiction. That means that the orthocenters are in fact aligned.

In particular the orthocentric line is the homothetical transformation of the Simson line with center point P and ratio 2.This means that the lines will be parallel, and that any perpendicular to one line will be perpendicular to the other. The pedal line **R** will go through the middle of a perpendicular from the focal point **P** to the orthocentric line **R'**. □

3.3. On the Gauss-Newton line. Giving answer to Question 6.

Many proofs have been given for Steiner's Question 6 (see [32], [33]). The question defines the line that goes through the midpoints of each diagonal. There are 3 diagonals on the complete quadrilateral, which are the lines joining opposite vertices. This line was called as Gauss-Newton Line, and the result was already known before Steiner proposed it as part of the questions. Some of the proofs were written in the Annales de Mathématiques Pures et Appliquées [34], with some more properties. Because of the interest to make an exhibit out of it, different options were considered.

One of the approaches to the question that appeared while working on it was to use vectors, some basic algebra and determinants. Although it is much different to the approach done in the other questions, I decided to write my own proof (very different to the ones cited) using these tools. The final reason was that these topics are familiar to students in High School, which are one of the possible demographicals groups of visitors for the exhibit.

Another way would be to use Gauss-Bodenmiller theorem, but this will be studied during Steiner's Question 7.

We will give a prove based on vectors, and linearity dependence and Menelaus Theorem. We will use Pedoe [35] notation and Lemmas on what are aligned points.

Given two points A and B, the **line** between A and B are the points $C = A + t\overrightarrow{AB}$ for $t \in \mathbb{R}$, considering $\overrightarrow{AB} = B - A$ we can rewrite it as $C = (1-t)A + tB$ for $t \in \mathbb{R}$, or in a similar way $C = xA + x'B$ for $x, x' \in \mathbb{R}$ and $x + x' = 1$.

Theorem 3.10. *If A, B and C are aligned points, then real numbers x, y, z not all zero, can be found such that $x + y + z = 0$ and $xA + yB + zC = 0$.*

Theorem 3.11. *If A, B and C are given points, and real numbers x, y, z not all zero, can be found such that $x + y + z = 0$ and $xA + yB + zC = 0$ then points A, B and C are aligned.*

Theorem 3.12. *If A, B and C are three given points which are not aligned, and we can find three real numbers x, y, z such that $x + y + z = 0$ and $xA + yB + zC = 0$, then we must have $x = y = z = 0$.*

Theorem 3.13 (Menelaus)**.** *Consider a triangle ABC and a line that intersects the sides AB, CA and BC in D, E and F (Figure 33) then the directed distances (and ratios) satisfy*

$$\frac{AD}{DB} \cdot \frac{BF}{FC} \cdot \frac{CE}{EA} = -1$$

.

Using Pedoe notation, if $D = xA + x'B$, $E = yC + y'A$, and $F = zB + z'C$, where $x + x' = y + y' = z + z' = 1$, then

$$xyz = -x'y'z'$$

An equivalent equation is:

$$1 - x - y - z + xy + yz + zx = 0$$

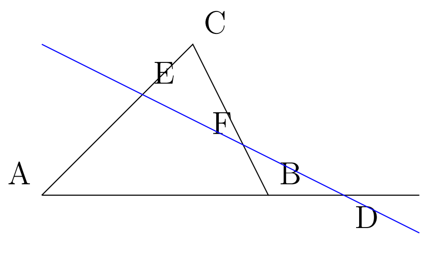


FIGURE 33. Menelaus configuration

Steiner's Question 6. The midpoints of the diagonals of the complete quadrilateral belong all three to the same line **R"** (Newton line) See Figure 34.

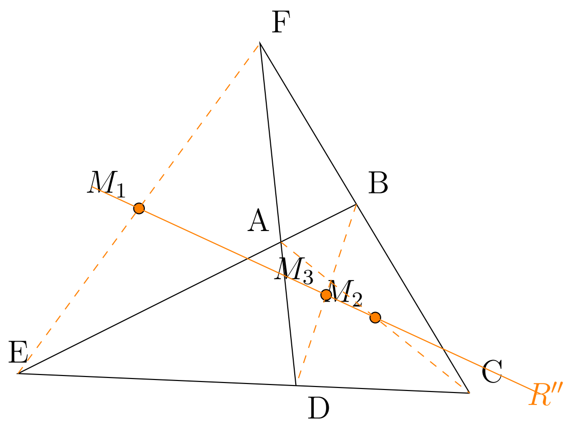


FIGURE 34. Steiner's Question 6

Proof. We will consider the complete quadrilateral as a triangle (three of the points) and a line that intersects the sides (the other three points). Let the triangle be FDC and let E, B, A be the intersection of the line with sides DC, CF and FD.

We consider now the midpoints of the diagonals. That is:

$$M_1 = \frac{1}{2}E + \frac{1}{2}F \quad M_2 = \frac{1}{2}A + \frac{1}{2}C, \quad M_3 = \frac{1}{2}B + \frac{1}{2}D$$

We want to see that these three points are aligned. Using Theorem 3.11 that means that $p + q + r = 0$ and $pM_1 + qM_2 + rM_3 = 0$ has a solution, $p, q, r \in \mathbb{R}$ not all 0.

Substituting M_1, M_2, and M_3 we have:

$$p(\frac{1}{2}E + \frac{1}{2}F) + q(\frac{1}{2}A + \frac{1}{2}C) + r(\frac{1}{2}B + \frac{1}{2}D) = 0$$

The restrictions on p, q and r, come from the fact that A $\in$ FD, B $\in$ CF and E $\in$ CD. Hence, $A = xF + x'D$ where $x + x' = 1$ $B = yC + y'F$ where $y + y' = 1$ and $E = zD + z'C$ where $z + z' = 1$. An extra restriction is that A $\in$ EB. This does not add any constrains as it only fixes the value of x (which was already fixed).

Substituting once more, we have:

$$p(\frac{1}{2}zD + \frac{1}{2}z'C + \frac{1}{2}F) + q(\frac{1}{2}xF + \frac{1}{2}x'D + \frac{1}{2}C) + r(\frac{1}{2}yF + \frac{1}{2}y'C + \frac{1}{2}D) = 0$$

Reorganizing terms we have:

$$(\frac{1}{2}pz + \frac{1}{2}qx' + \frac{1}{2}r)D + (\frac{1}{2}pz' + \frac{1}{2}q + \frac{1}{2}ry)C + (\frac{1}{2}p + \frac{1}{2}qx + \frac{1}{2}ry')F = 0$$

Now, we can see that the coefficients of D, C and F add up to 0:

$$(\frac{1}{2}pz + \frac{1}{2}qx' + \frac{1}{2}r) + (\frac{1}{2}pz' + \frac{1}{2}q + \frac{1}{2}ry) + (\frac{1}{2}p + \frac{1}{2}qx + \frac{1}{2}ry') =$$

$$= (\frac{1}{2}pz + \frac{1}{2}pz' + \frac{1}{2}p) + (\frac{1}{2}qx' + \frac{1}{2}q + \frac{1}{2}qx) + (\frac{1}{2}r + \frac{1}{2}ry + \frac{1}{2}ry') = p + q + r = 0$$

since $x + x' = 1$, $y + y' = 1$ and $z + z' = 1$. By Theorem 3 we have that the only solution will be that each coefficient is 0, as we know that D, F and C are not aligned.

This gives us the system: $\begin{cases} \frac{1}{2}pz + \frac{1}{2}qx' + \frac{1}{2}e = 0 \\ \frac{1}{2}pz' + \frac{1}{2}q + \frac{1}{2}ey = 0 \\ \frac{1}{2}p + \frac{1}{2}qx + \frac{1}{2}ey' = 0 \\ p + q + r = 0 \end{cases}$

Because the system is homogeneous we know that it will not be inconsistent, the trivial solution $p = 0, q = 0$ and $r = 0$ satisfies the constrains. The points will be aligned if we prove that there are more solutions. In this case by Rouché–Frobenius theorem the rank of the matrix has to be less than the number of variables (3). We have to check four determinants.

$$\begin{vmatrix} \frac{1}{2}z & \frac{1}{2}x' & \frac{1}{2} \\ \frac{1}{2}z' & \frac{1}{2} & \frac{1}{2}y \\ \frac{1}{2} & \frac{1}{2}x & \frac{1}{2}y' \end{vmatrix}, \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{2}z' & \frac{1}{2} & \frac{1}{2}y \\ \frac{1}{2} & \frac{1}{2}x & \frac{1}{2}y' \end{vmatrix}, \begin{vmatrix} \frac{1}{2}z & \frac{1}{2}x' & \frac{1}{2} \\ 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2}x & \frac{1}{2}y' \end{vmatrix}, \begin{vmatrix} \frac{1}{2}z & \frac{1}{2}x' & \frac{1}{2} \\ \frac{1}{2}z' & \frac{1}{2} & \frac{1}{2}y \\ 1 & 1 & 1 \end{vmatrix}.$$

We will only compute the first one and the second one, because the last two are similar to the second one.

$$\begin{vmatrix} \frac{1}{2}z & \frac{1}{2}x' & \frac{1}{2} \\ \frac{1}{2}z' & \frac{1}{2} & \frac{1}{2}y \\ \frac{1}{2} & \frac{1}{2}x & \frac{1}{2}y' \end{vmatrix} = \frac{1}{8}zy' + \frac{1}{8}x'y + \frac{1}{8}z'x - \frac{1}{8}zyx - \frac{1}{8} - \frac{1}{8}x'y'z' =$$

$$= \frac{1}{8}z(1-y) + \frac{1}{8}(1-x)y + \frac{1}{8}(1-z)x - \frac{1}{8}zyx - \frac{1}{8} - \frac{1}{8}x'y'z' =$$

$$= \frac{1}{8}z - \frac{1}{8}zy + \frac{1}{8}y - \frac{1}{8}xy + \frac{1}{8}x - \frac{1}{8}zx - \frac{1}{8}zyx - \frac{1}{8} - \frac{1}{8}x'y'z' = 0$$

This last step uses Menelaus theorem in both forms stated after Theorem 3.13. $xyz = -x'y'z'$ and $1 - x - y - z + xy + yz + zx = 0$

The second determinant:

$$\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{2}z' & \frac{1}{2} & \frac{1}{2}y \\ \frac{1}{2} & \frac{1}{2}x & \frac{1}{2}y' \end{vmatrix} = \frac{1}{4}y' + \frac{1}{4}y + \frac{1}{4}z'x - \frac{1}{4} - \frac{1}{4}xy - \frac{1}{4}z'y' =$$

$$= \frac{1}{4} + \frac{1}{4}z'x - \frac{1}{4} - \frac{1}{4}xy - \frac{1}{4}z'y' = \frac{1}{4}z'x - \frac{1}{4}xy - \frac{1}{4}z'y' =$$

$$= \frac{1}{4}z'x - \frac{1}{4}xy - \frac{1}{4}(1-z)y' = \frac{1}{4}z'x - \frac{1}{4}xy - \frac{1}{4}y' + \frac{1}{4}zy' = \frac{1}{4}(1-z)x - \frac{1}{4}xy - \frac{1}{4}(1-y) + \frac{1}{4}z(1-y) =$$

$$= -\frac{1}{4} + \frac{1}{4}x + \frac{1}{4}y + \frac{1}{4}z - \frac{1}{4}xy - \frac{1}{4}xz - \frac{1}{4}zy = 0$$

Again the last step is an implication of Menelaus. This means that $p = 0, q = 0$ and $r = 0$ is not the only solution. Because there is more than one solution (not all of them zero), by Theorem 3.11 we can conclude that M_1, M_2 and M_3 are aligned. □

3.4. **On the power of a point with respect to a circle. Giving answer to Steiner's Question 7.**

It was in 1826 when Jakob Steiner defined the concept of the power of a point P with respect to a circle C [31]. This concept is going to be key in the proof we will explore for Steiner's Question 7.

Definition 3.14 (Power of a point P with respect to a circle C)**.** Consider a point P and a circle C with center O, and consider a line that intersects the circle (at points D and E). Then the product PD·PE is constant and does not depend on the choice of the line. It is called the Power of P with respect to C and it is written by $Power(P, C)$.

In particular Figure 35 shows the three cases:

- Intersecting general points D and E. $Power(P, C) = PD \cdot PE$.
- Consider the line that goes through the center of C, and let d be the distance PO and r the radius of C, then
  $Power(P, C) = PA \cdot PB = (d - r)(d + r) = d^2 - r^2$.
- Consider a tangency point T from P to C. Now, $Power(P, C) = PT^2$.

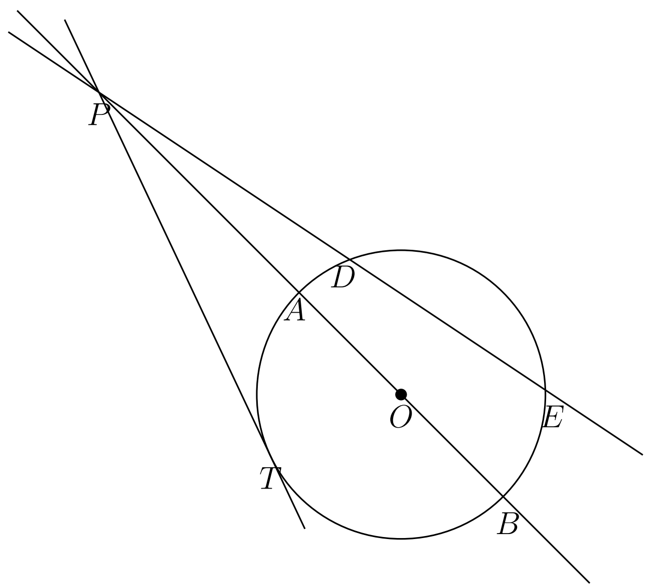


FIGURE 35. Power of a point with respect to a circle

Definition 3.15 (The radical axis of two circles)**.** The loci of points that have the same power with respect to two circles is called *radical axis* (see Figure 36).

Some basic properties (proved in Pedoe [35]) are:

Property 3.16. The radical axis is a line (even if the circles do not intersect). In case they intersect, the radical axis is the line obtained by joining such intersections.

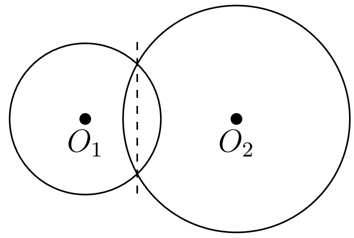


FIGURE 36. Radical axis of two intersecting circles

A special case is found when the circles are tangent. In such a case it is the tangent to both circles at the tangency point.

Property 3.17. The radical axis is perpendicular to the line that connects the centers of both circles.

Property 3.18. Consider two circles C_1, C_2 and their radical axis r, then if we create a third circle that intersects C_1 and C_2, the intersections of the lines joining the intersections belong to r. Figure 37 shows this property.

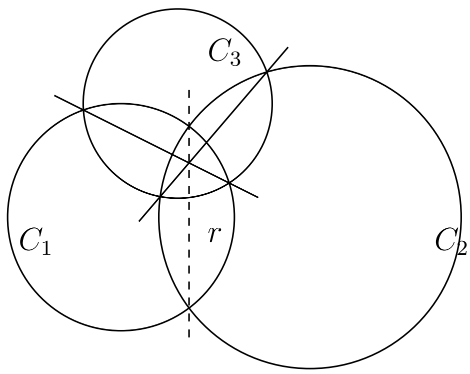


FIGURE 37. The intersection of chords from a third circle belong to the radical axis

The main result that allows us to prove Steiner's Question 7 and that was not really stated by Steiner is the Gauss-Bodenmiller theorem [30].

Theorem [Gauss-Bodenmiller]: The orthocentric line of a complete quadrilateral is the radical axis of the circles with diagonals for diameters (see Figure 38). Notice that although there are three diagonals there is only one radical axis (it is the same when taken the circles pairwise).

Proof. Let C_{EF}, C_{AC} and C_{BD} be circles with diameters EF, AC, and BD. Let M_1, M_2 and M_3 be the midpoints of these diagonals which are the centers of the three circles above.

We will show that the orthocenters belong to the radical axis. Following Figure 39A, let us focus in AED. Drawing the circle with diameter ED, we can see it intersects EA in D' and AD in E'. Because the central angle that covers the same arc as EE'D and ED'D is π then the lines DD' and EE' are perpendicular to the sides AE and AD, which means that they are two of the heights of the triangle AED. The intersection will be its orthocenter. By property 3.18 this orthocenter will belong to the radical axis of the circle build with EF as diameter, and the circle with diameter

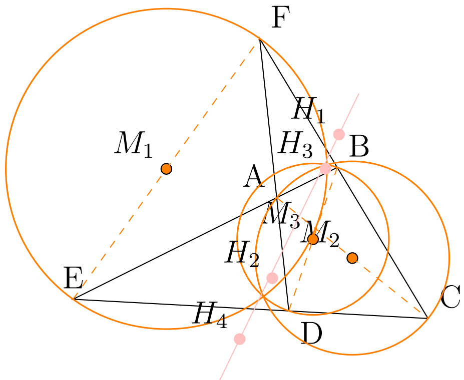


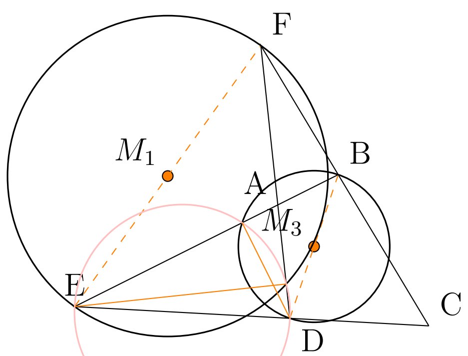
FIGURE 38. Orthocenter of triangle AED is in the radical axis of circles with diamaters EF and AC

BD. Because of analogy of arguments the same reasonment can be used with triangle ABF. This means that the radical axis contains two of the orthocenters. Since Steiner's Question 4 concluded that the orthocenters were aligned in the orthocentric line, if it includes two it will include the whole line, as there is a unique line through two points. There is though a particular case were this wouldn't work and it occurs when $FD \perp BE$. In this case the circles would be tangent, and the orthocenters of both triangles would be the same point A. The same reasoning could be done with triangle EBC.

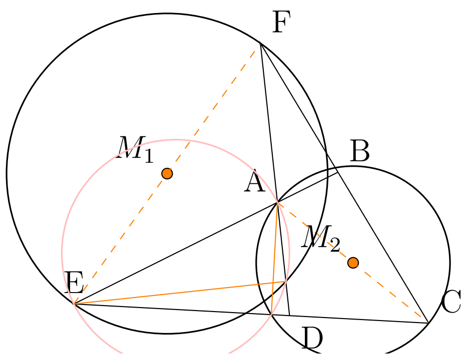
Figure 39B shows a similar position for C_{AC} and C_{BD}, with it we can conclude that is also the radical axis of these two. Because it is the radical axis of the two pairs, it will be the radical axis of the 3 circles.

There is are some important remarks to make here:

- We could have done the same reasoning for the different triangles, and thus not using that the points are aligned, being able to use this theorem then to prove Steiner's Question 4.
- We could use this to prove that the midpoints are aligned. Taking pairs of midpoints (centers of circles) would imply the lines would be perpendicular to the orthocentric line and thus all 3 belonging to the same line.



(A) Does the orthocenter AED belongs to the radical axis?



(B) Another orthocenter to be analysed

FIGURE 39. Orthocenters belong to the radical axis

□

Steiner's Question 7. Line **R"** is a common perpendicular to both the pedal line **R** and orthocentric line **R'** (See Figure 40).

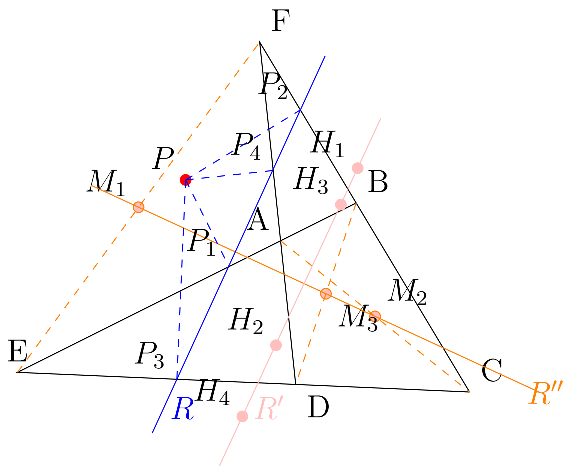


FIGURE 40. Steiner's Question 7

Proof. This is an immediate consequence of the Gauss-Bodenmiller theorem. The radical axis of two circles is perpendicular to the line that joins its centers (by property 3.17), which means that the orthocentric line is perpendicular to the line that contains the midpoints of each diagonal. This midpoint line being perpendicular to Simson's line is due to Steiner's Question 5, and the parallelism between pedal line and the orthocentric line. □

3.5. **On incircles and excircles. Proving Steiner's Questions 8 and 9.**

Although we are arriving at the end of the theorems, for these two questions only the construction of the incircle and the excircles along with lemmas 3.2 and lemma 3.6 are needed.

The incenter of a triangle ABC is obtained when drawing the interior angle bisectors. For each vertex we have two bisectors, an interior and an exterior one. These two lines are perpendicular to each other. The excircles are constructed by intersecting two of the exterior bisectors, and an interior one (although the intersection will be outside of the triangle).

Steiner's Question 8. For each of the four triangles of the complete quadrilateral consider its inscribed circle and three excircles, which makes in total *sixteen* circles. Then the centers of these sixteen circles are four by four in the same circle, creating *eight* new circles (see Figure 41).

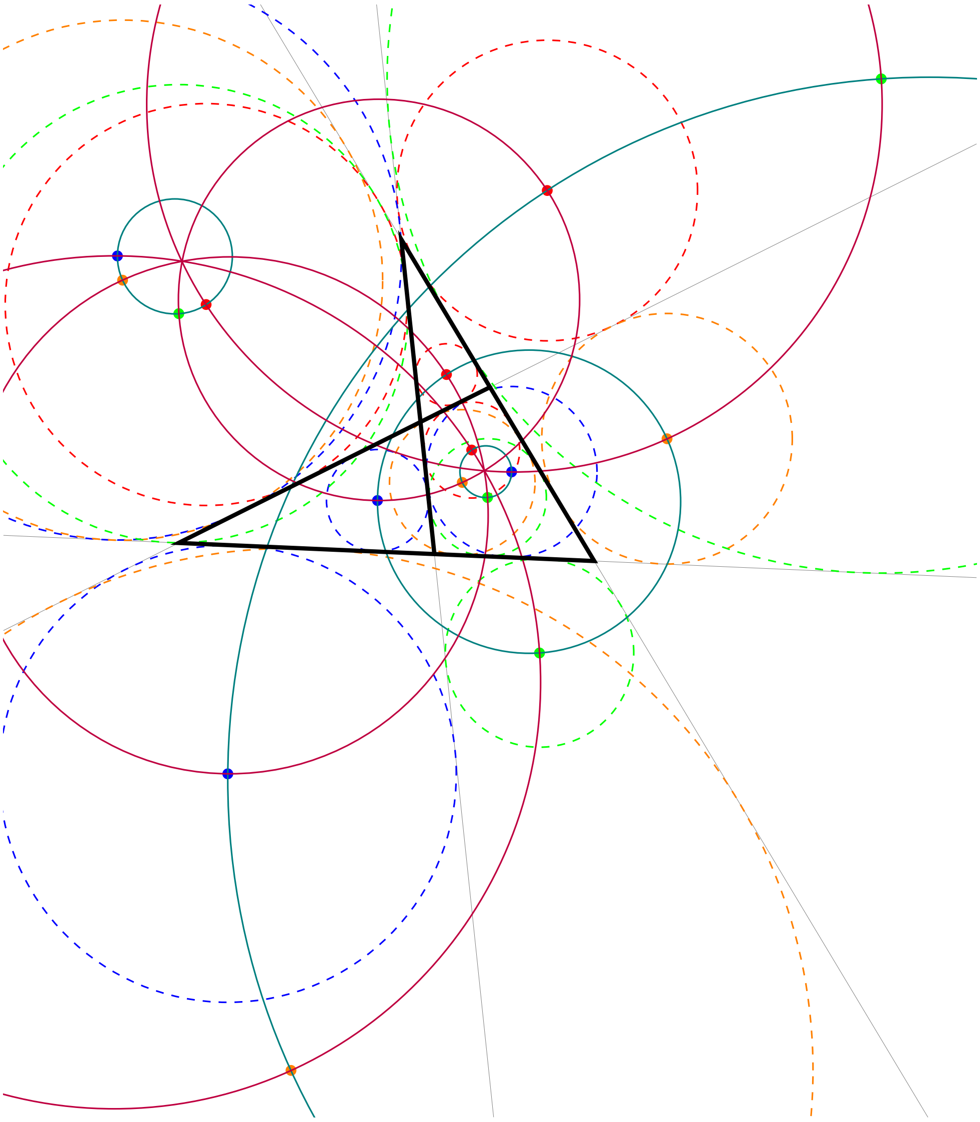
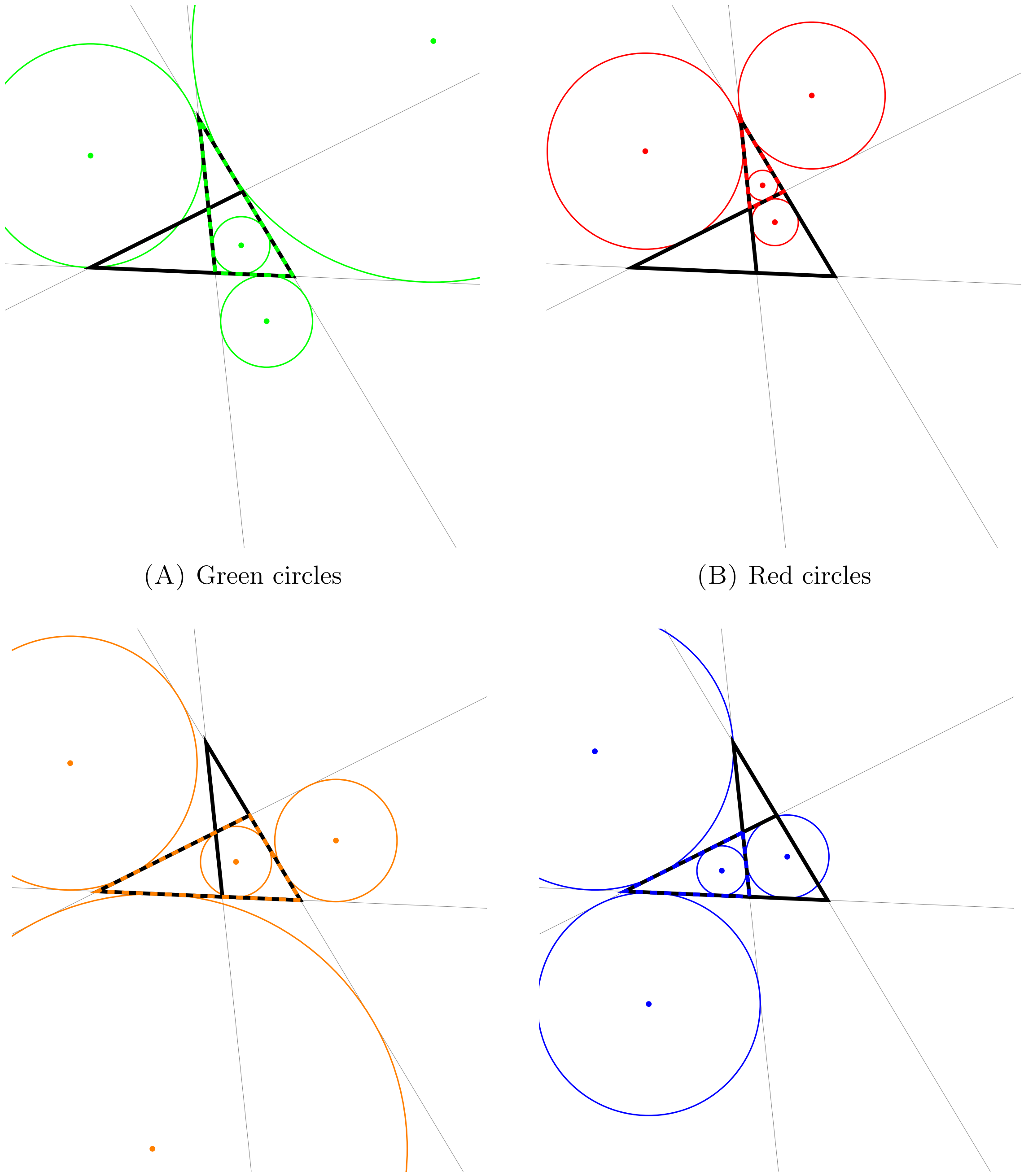


FIGURE 41. Steiner's Question 8

Before going to the proof we will show the incircles and excircles in separate images (see Figure 42), to help understand what is happening. We will also show the 8 new circles. Then we will present two cases of proofs where we will apply angle chasing to solve the questions.



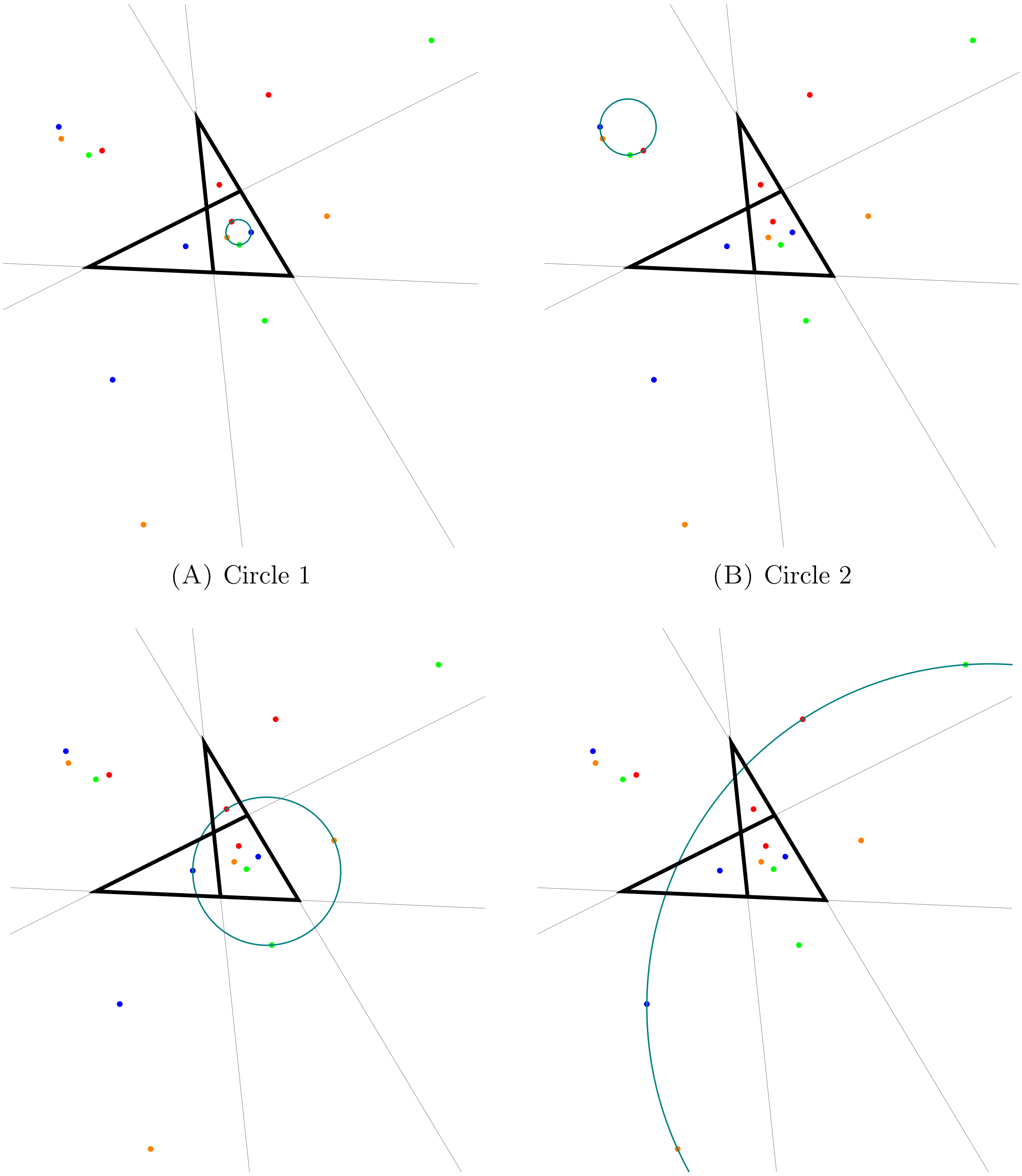
(A) Green circles

(B) Red circles

(C) Orange circles

(D) Blue circles

FIGURE 42. Inscribed / escribed circles



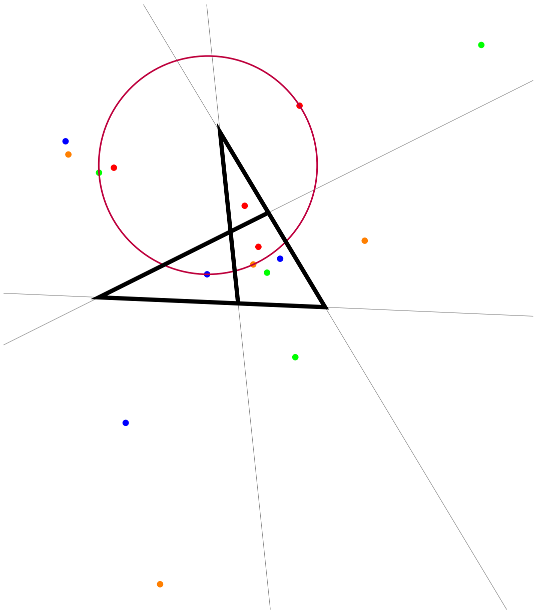
(A) Circle 1

(B) Circle 2

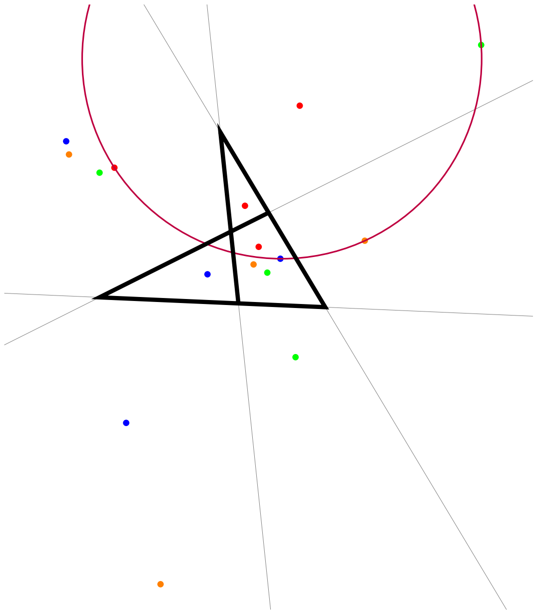
(C) Circle 3

(D) Circle 4

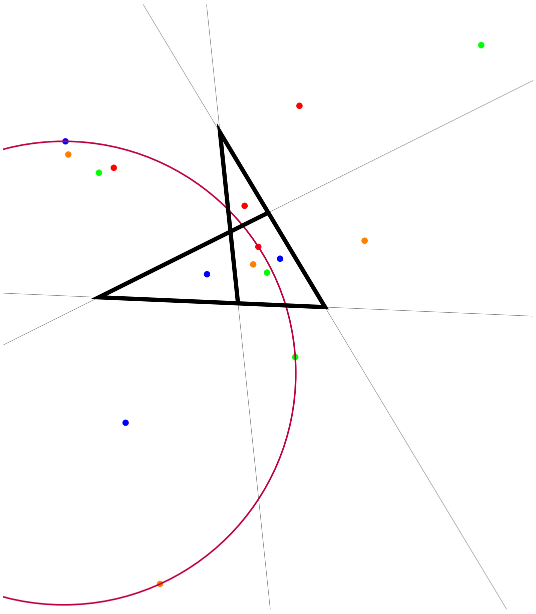
FIGURE 43. Circle of centers (Teal family)



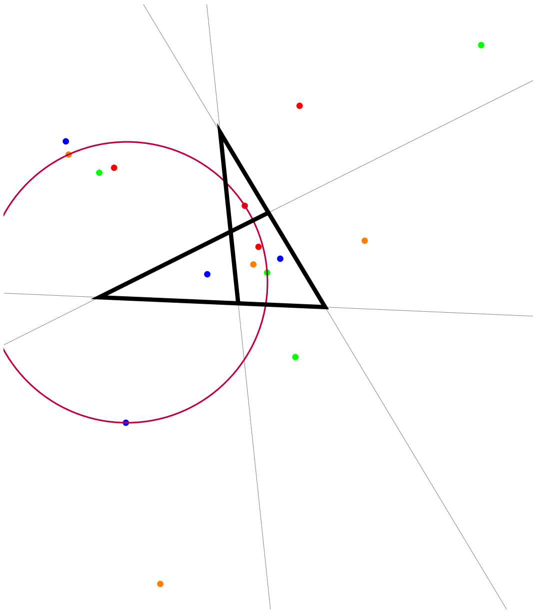
(A) Circle 5



(B) Circle 6



(C) Circle 7



(D) Circle 8

FIGURE 44. Circle of centers (Purple family)

To be able to show that the incenters are concyclic we will give two different proofs depending on the angle bisectors used. The first proof will be based on the *inscriptible quadrilateral* lemma (3.6) and the second on the *angles covering same arc* Corollary 2 in Lemma (3.2).

Figure 45 corresponds to the third circle in the previous images. We will prove that the centers concyclic.

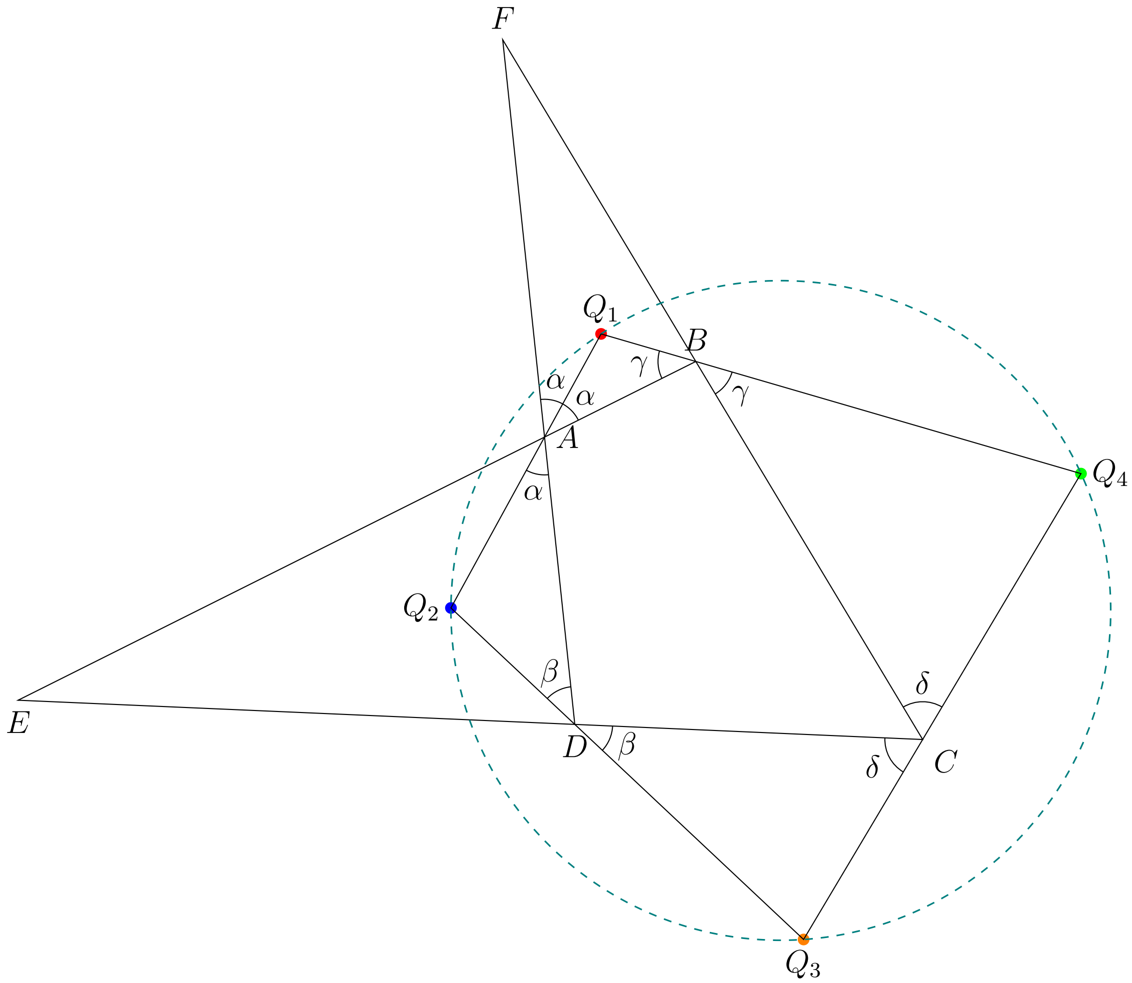


FIGURE 45. Case 1

Proof. We will use $\hat{Q}_i$ to indicate the interior angle of the quadrilateral at Q_i. To see it is inscriptible we need to see that $\hat{Q}_1+\hat{Q}_3=\pi$.

Let α be the angle $\angle BAQ_1$. Because Q_1, the incenter, is build using the angle bisectors. That means $\angle BAQ_1 = \angle Q_1AF = \angle Q_2AD$, since $\angle Q_1AF$ and $\angle Q_2AD$ are opposite to each other. The same goes for the rest of the angles β, δ and γ.

Writing the angles of the quadrilateral in terms of these angles we have:

$$\hat{Q}_1 = \pi - \alpha - \gamma, \quad \hat{Q}_2 = \pi - \alpha - \beta, \quad \hat{Q}_3 = \pi - \beta - \delta, \quad \hat{Q}_4 = \pi - \delta - \gamma$$

.

Since a quadrilateral could be divided in two triangles, we know that the sum of their angles is always 2π.

Writing the sum and rearranging:

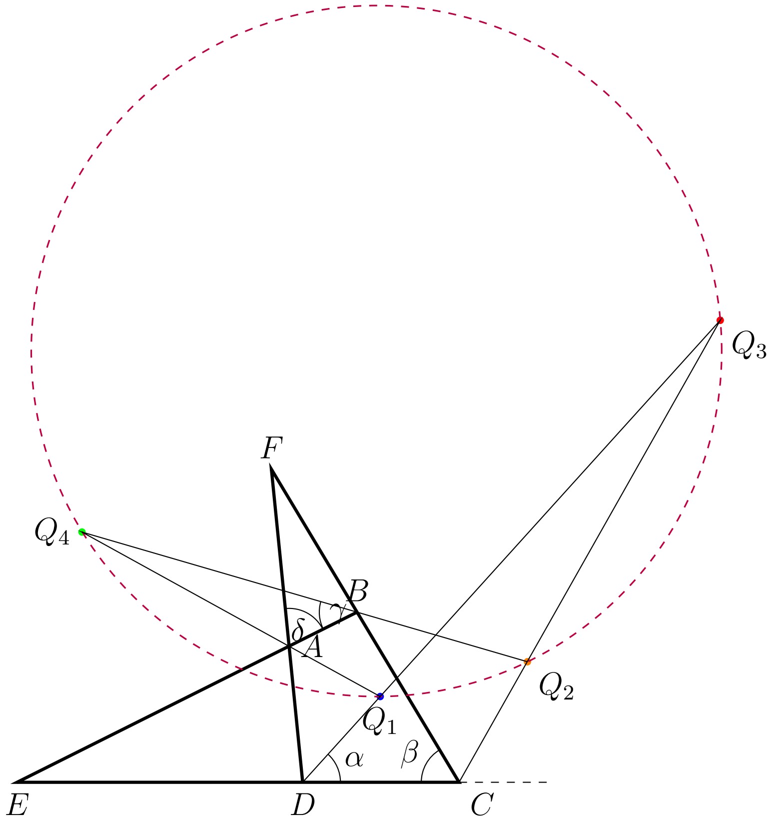


FIGURE 46. Case 2

$$\hat{Q}_1 + \hat{Q}_2 + \hat{Q}_3 + \hat{Q}_4 = 2\pi - 2\alpha - 2\gamma + 2\pi - 2\beta - 2\delta = 2\hat{Q}_1 + 2\hat{Q}_3 = 2\pi$$

That means $\hat{Q}_1 + \hat{Q}_3 = \pi$. By lemma 3.6 the quadrilateral is inscriptible and thus Q_1, Q_2, Q_3 and Q_4 are concyclic. □

A second case will be studied, which corresponds to circle 6. In this case we can see that some of the centers are on the same side of the angle bisector. We can see in Figure 46 that Q_1 and Q_3 are on the same side as the angle bisector at D, while Q_1 and Q_4 are on opposing sides of bisectors at A.

Proof. Using the notation in Figure 46 we can see that the points are concylic by checking that $\angle Q_1Q_3Q_2 = \angle Q_1Q_4Q_2$.

We can see that $\angle Q_3CD = \frac{\pi-\beta}{2} + \beta = \frac{\pi}{2} + \frac{\beta}{2}$ then $\angle DQ_3C = \pi - \alpha - \frac{\pi}{2} - \frac{\beta}{2}$

Similarly $\angle BAQ_4 = \frac{\pi-\delta}{2} + \delta$ then $\angle AQ_4B = \pi - \gamma - \frac{\pi}{2} - \frac{\delta}{2}$

Using $\angle DFC = \angle AFC$ we have $\pi - 2\alpha - \beta = \pi - \delta - 2\gamma$ Then, -$\alpha - \frac{\beta}{2} = -\gamma - \frac{\delta}{2}$

Which implies $\angle DQ_3C = \angle AQ_4B$, so $\angle Q_1Q_4Q_2 = \angle Q_1Q_3Q_2$. Thus the points Q_1, Q_2, Q_3 and Q_4 are concyclic.

The other cases of circles can be solved in a similar manner than this two cases using the proper angles and lines.

□

Steiner's Question 9. These new eight circles can be divided in two groups such that each of the four circles in one these groups intersects orthogonally all the circles of the other group; we can conclude that the centers of the circles of both groups belong to two lines one perpendicular to the other (see Figure 47). These lines are called the *incentric lines* of the complete quadrilateral.

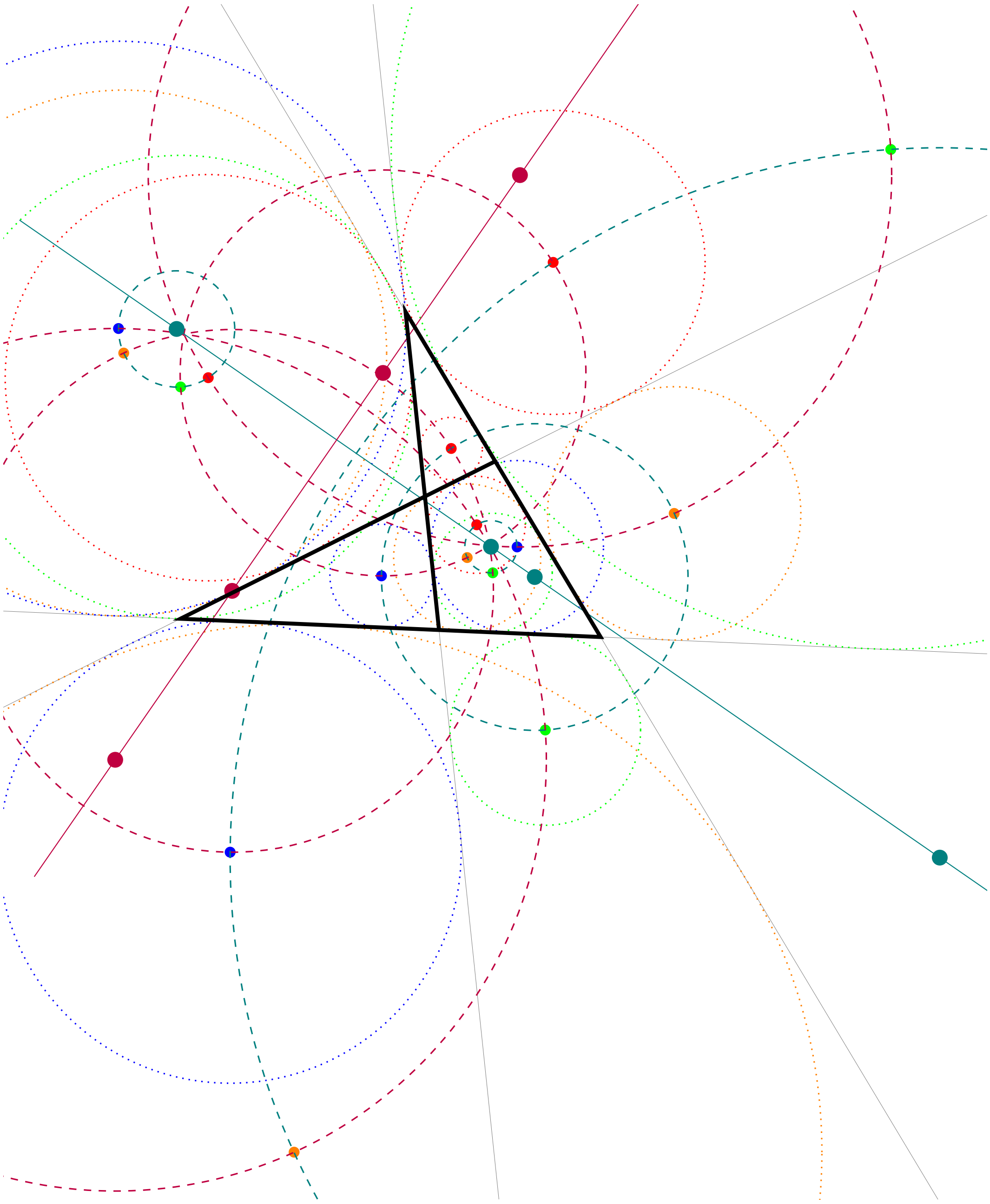


FIGURE 47. Steiner's Question 9

Proof. Another angle chasing can help us see that the circles of the two families are orthogonal to each other. Consider for example two of the circles with centers C_Q and C_I (see Figure 48) build using the incenters and excenters Q_1, Q_2, Q_3, Q_4, I_1, I_2, I_3, I_4. Note that between two of the circles (one of each family) we can always

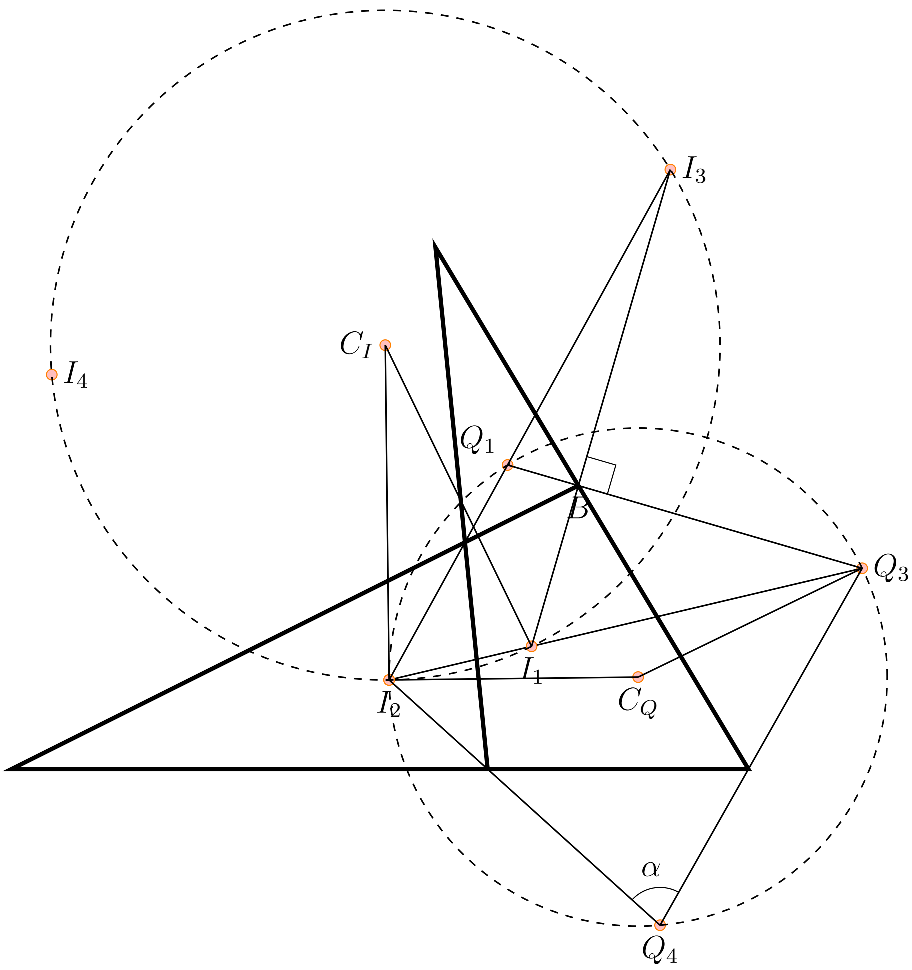


FIGURE 48. Important lines for the proof

find a common point which is a center (incenter or excenter) on itself. In this case $Q_2 = I_2$. Let α be the angle $\angle Q_3Q_4I_2$.

As Q_1 is opposite of Q_4 in the cyclic quadrilateral, then $\angle I_2Q_1Q_3 = \pi - \alpha$.

Using this and observing that Q_1, I_2, I_3 belong to the same bisector, $\angle Q_3Q_1I_3 = \alpha$.

Thus Q_1Q_3 and I_3I_1 are both angle bisectors at B, and thus I_3BQ_1 is right.

$\angle Q_1I_3B = \pi - \alpha - \frac{\pi}{2} = \frac{\pi}{2} - \alpha$.

The central angle that covers the same arc is double. $\angle I_2C_II_1 = 2\angle I_2I_3I_1 = \pi - 2\alpha$.

Triangle $I_2I_1C_I$ is isosceles, $2\angle I_1I_2C_I + \angle I_2C_II_1 = \pi$, thus $\angle I_1I_2C_I = \alpha$.

On the other hand:

$\angle Q_3C_QI_2 = 2\alpha$ since it is the central angle to the inscribed $\angle Q_3Q_4I_2$.

Then $I_2C_QQ_3$ being isosceles, $2\angle C_QI_2Q_3 + \angle Q_3C_QI_2 = \pi$,

So $\angle C_QI_2Q_3 = \frac{\pi}{2} - \alpha$. Finally,

$$\angle C_QI_2C_I = \angle C_QI_2Q_3 + \angle Q_3I_2C_I = \frac{\pi}{2} - \alpha + \alpha = \frac{\pi}{2},$$

which is what we wanted to prove. That the radius at the intersection are perpendicular and thus the circles are orthogonals.

Second part of Steiner's Question 9 states that using the orthogonality we must conclude that the centers are aligned.

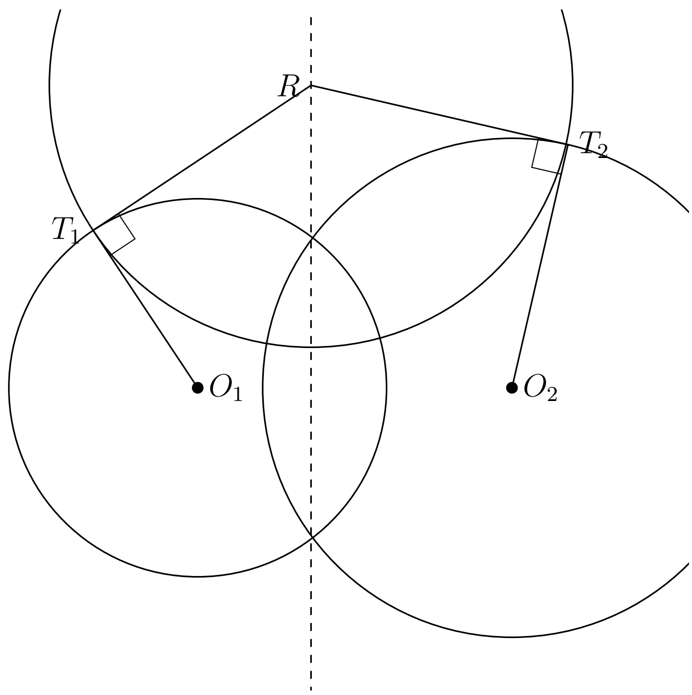


FIGURE 49. Orthogonal circles to 2 given circles have the center in the radical axis.

Given two circles C_1 and C_2, the loci of possible centers for orthogonal circles is the radical axis between the two circles (see Figure 49). Imposing that the third circle has to be orthogonal to the previous two, we can draw tangents from the points of intersections T_1 and T_2 which intersect each other at a certain point R. The third circle will be at a distance r from T_1 and T_2, then $Power(R, C_1) = r^2 = Power(R, C_2)$. This means that the power is the same which means R belongs to the radical axis.

Having show that the circles are orthogonal we need to see that the centers in each family are aligned and generate two perpendicular lines.

Let C_1, C_2, C_3 and C_4 be one of the families, and C_5, C_6, C_7 and C_8 be the other. We know C_1 is orthogonal to C_5 and C_6 that means that that the center of C_1 belongs to the radical axis of C_5 and C_6. Because the radical axis is fixed for them, the circles C_2, C_3 and C_4 will also belong to it (as they are orthogonal as well). This means that all the centers will be aligned. A similar reasoning can put C_5, C_6, C_7 and C_8 on the radical axis of C_1, C_2, and that means that they are also aligned. Finally these two lines are perpendicular, as it is a property of the radical axis and the line joining the centers (that both are perpendiculars). □

3.6. **On the circle of nine points. Proving Steiner's Question 10.**

Theorem 3.19 (Nine-point circle[35]). *Given a triangle ABC, and let M_{AB}, M_{BC}, M_{AC} be the midpoints of each side. Let H be the orthocenter and H_A, H_B and H_C be the feet of each height. Then there exists a circle that goes through M_{AB}, M_{BC}, M_{AC}, H_A, H_B, H_C and through the midpoints of the segments AH, BH and CH. This circle is called the nine-point circle (see Figure 50).*

Steiner's Question 10. Finally the incentric lines, from Steiner's Question 9, meet at point the focal point **P** defined in Steiner's Question 1 (see Figure 51).

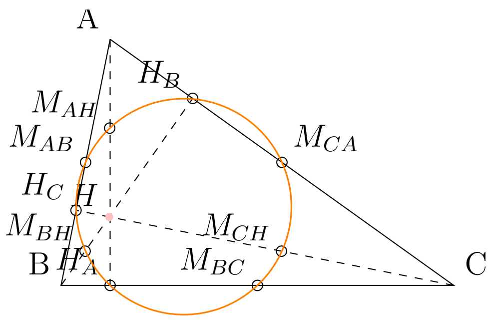


FIGURE 50. Nine point circle

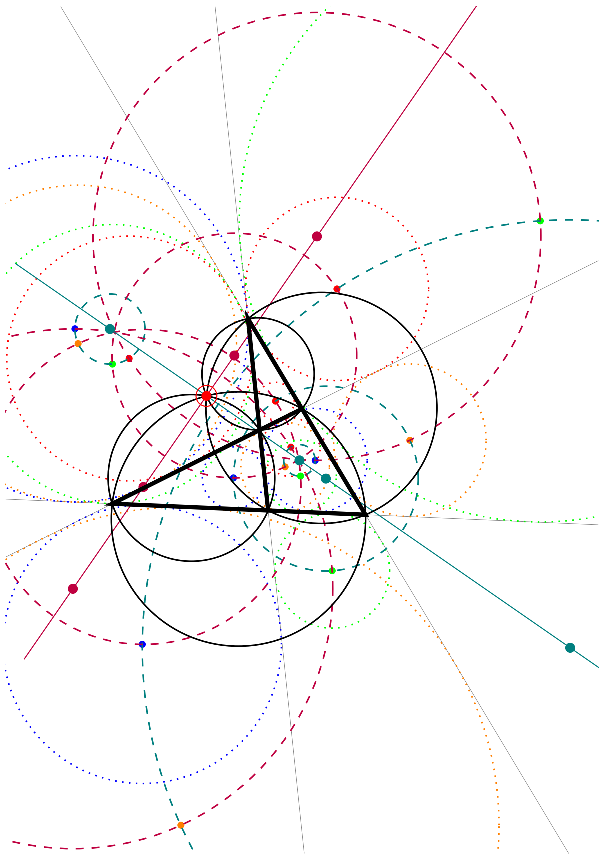
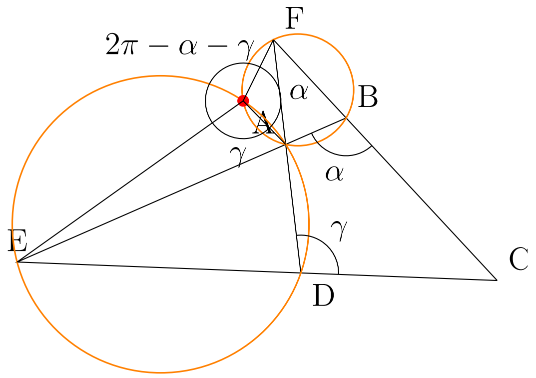


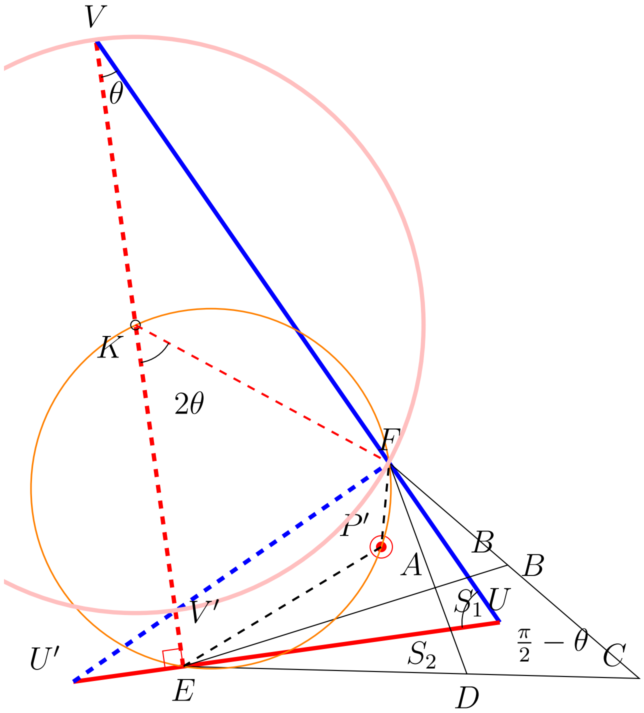
FIGURE 51. Steiner's Question 10

The following proof is based on the one Ehrmann writes in his paper [36], but adapting the notation, and using angle chasing. Other proofs can be found in [23], [37], [26], [24].

Proof. The outline of the proof will be as follows. We will use a pair of opposite vertex of the complete quadrilateral and we are going to draw their angle bisectors. The configuration with the four angle bisectors will allow us to describe a circle of nine points. We will then prove that the focal point of the complete quadrilateral belongs to this nine point circle. We will also show that this circle contains the intersection of the two incentric lines. Finally repeating this process for the other pairs of opposite vertexes, we will be able to conclude that the intersection of the incentric lines is in fact the focal point of the quadrilateral.



(A) Rewritting $\angle FPE$



(B) $\angle FP'E$

FIGURE 52. Studying P and P'

Following Figure 52. Let UU' and VV' be the angle bisectors at vertex E, and let UV and V'U' be the bisectors at vertex F. Let K be the midpoint of segment VV'. The circle of nine points of U'VU contains K, E and F, since $EV \perp UU'$ and $U'F \perp VU$ and K is the midpoint between the orthocenter V' and the vertex V. We will prove that the angle $\angle FP'E =$$2\pi - EPF$.

Following Steiner's Question 1 we can see that $\angle EPF = 2\pi - \angle CDF - \angle EBC$ (see Figure 52). We will prove that it is the same angle than $\angle EP'F$.

Let $\angle FVK = \theta$. The central angle $\angle FKV' = 2\theta$ and $\angle VUE = \frac{\pi}{2} - \theta$.

$$\angle US_2F = \angle ES_2D = \pi - \angle DES_2 - (\pi - \gamma) = \gamma - \angle DES_2$$

$$\angle US_1F = \alpha - \angle AFS_1$$

Adding the angles of the triangles FS_2U and ES_1U we obtain:

$$\angle EUS_1 + \angle US_1E + \angle S_1EU + \angle FUS_2 + \angle US_2F + \angle S_2FU = 2\pi$$

Substituting we obtain:

$$2\theta = \alpha + \gamma - \pi$$

We can see that

$$\angle EKF + \angle FP'E = \pi$$
$$\angle FP'E = \pi - (\alpha + \gamma - \pi) = 2\pi - \alpha - \gamma$$

We can see this is the angle of $\angle FPE$, which means P belongs to the nine point circle.

On the other hand, let Q be the midpoint of VU' and Q' the midpoint of V'U. Then these two points belong to the circle of nine points. Also we will use that Q belongs to the incentric line (the line from Steiner's Question 8) and Q' belongs to the other incentric line [36], and knowing that these lines are perpendicular (Steiner's Question 8), then we know that the intersection will belong to a circle with diameter QQ'.

The line QQ' is obtained by joining the midpoints of the diagonals of the complete quadrilateral V, V', U', E, U, F. By the result in Steiner's Question 7, this will be perpendicular to the Orthocentric line. Both F and E are orthocenters of the triangles of that quadrilateral. That means that QQ' *bot* EF. Knowing that the E and F are at the same distance of Q (and also at the same distance of Q'), we can conclude that QQ' is a diameter of the circle of nine points.

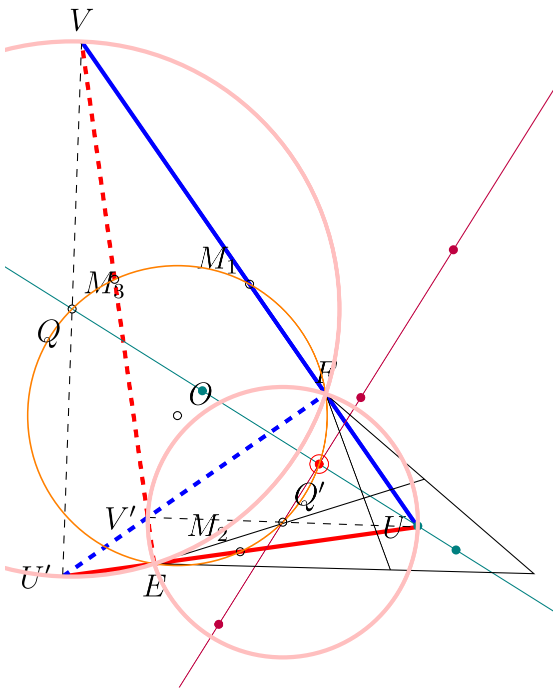


FIGURE 53. The nine-point circle

Repeating the same process for the vertices A, C and the vertices B, D, we can see that the intersection of the incentric lines belongs to the circles PED, ACP and BDP. These circles are not coaxial (the centers are not aligned) and thus the only common point is P. This means that the focal point P from Steiner's Question 1 is is the intersection of the incentric lines from Steiner's Question 9.

□

With this last proof, we conclude Steiner's 10 Questions on the complete quadrilateral.

4. Creation of the exhibit

4.1. **Vision and Mission of the exhibit.** At the start of this paper the only restriction on the creation of the exhibit was its content: Steiner's 10 questions on the complete quadrilateral (Appendix A contains the images of the questions, to accompany this proposals) Nonetheless the mission and vision of the exhibit were not defined.

In 1827 Jakob Steiner published in the Annales Mathématiques pures et appliqués 10 questions without answer on the complete quadrilateral. His aim was to challenge young mathematicians to solve them.

The nature of the theorems (Geometry) make it perfect for a visual exhibition. The level required to understand the questions is not difficult, but the level required to prove some of the statements is not easy either. In fact geometry has been through out history a good example of the meaning of mathematics, it has an abstract part, an applied one, proofs, conjectures, and it is easy to present specific cases where the mathematician needs to solve a specific problem.

We have decided to focus the exhibit on the space that current science museums do not cover.

Some of the options studied as goals for the exhibit then are as follow:

- Explain Steiner's 10 Questions
- Explain how mathematician work with abstract elements instead of specific cases
- Give some examples of Mathematical proofs
- Show different ways of Mathematical communication
- Design an exhibit for adults/experts on the context

Some of these options are in contradiction to the theoretical approach that experts have on the museization of science [3]. Maybe this is the reason why there is a lack of exhibits that cover this goals. For this reason and because some of the goals might contradict each other, we will present 2 proposals of exhibit. Both exhibits will work around the first goal, the *complete quadrilateral* and Steiner's 10 Questions, but one option will be designed for a Science Museum, with a focus on children, and without previous knowledge required, while the other proposal will be an independent exhibit for (or graduate or undergraduate students).

4.2. **Proposal 1: "What are Mathematics?".**

4.2.1. *Concept.* Mathematics have been miss-conceptualized for a long time. This exhibit will offer visitors an approach to what are using Steiner's 10 Questions as a common link through the exhibit.

Mathematics are the formal study of patterns. Problem solving, rigour, logic and proofs are the base of mathematics [38].

This exhibit is designed to be a family friendly exhibit to introduce the idea of mathematics to children. The visitor does not need to have any previous knowledge, and the aim of the exhibit will be to inspire some questions, and to give some (few) answers. At the end of the exhibit more challenges can be taken home.

4.2.2. *Content.* The exhibit consists of 9 displays. The displays are summarized in Table 1. Each display will occupy a specific space from the exhibit hall. A possible distribution is given in Figure 54.

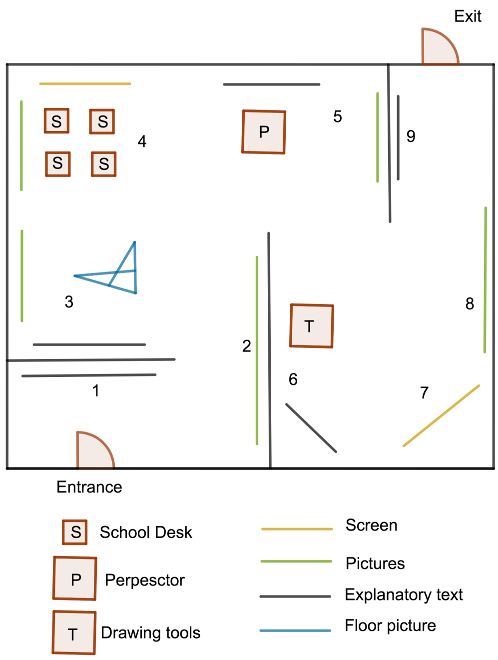


FIGURE 54. Imaginary distribution of the exhibit

FIGURE 55. Jakob Steiner

- **Display 1: The challenging Jakob Steiner**
  Text + Image "Steiner (1796-1863) was a Swiss mathematician that proposed in 1827 ten questions around the complete quadrilateral. He wrote them in Annales de Mathematiques pures et apliquées, without giving the answers. Some mathematicians have tried to solve them, some have succeded, some haven't. With time Steiner's 10 Questions were solved. This exhibit presents through this 10 Questions, what mathematics are, and how are they communicated. Enjoy!" Figure 55 would be included.
- **Display 2: The wall of possible truths**
  Text + Images (30 Images in a 6x5 Layout) The aim of mathematics is not only to give conjectures of facts, but to prove them. There will be a wall of 6 x 5 images that asks questions and that the visitor has to think if they are always true or not. These images give some theoretical background to the mathematics needed for the rest of the exhibit. An example of the questions:

Display	Title	Concept
1	The challenging Jakob Steiner	Introduction to the exhibit. Explanation of Steiner and the 10 Questions.
2	The wall of possible truths	Layout of 30 turning pieces that pose questions to the visitor. They contain basic concepts and definition of complete quadrilateral.
3	A quadrilateral on the floor	Quadrilateral drawn on the floor to discuss the construction of Steiner's Questions 1 and 2. Explanations on the wall panels.
4	The classroom	Black Board vs. Smart board. Experimenting on the two learning tools explore mathematics. Questions 3, 4 and 5 displayed.
5	Mathematics or art?	Wooden perspective frame, and complete quadrilaterals in perspective. Question 6.
6	Drawing in the past...	Learning about drawing tools. Question 7 on the table. Question 8 on an architecture table.
7	Drawing in the present...	Computer assisted drawing. Movement and construction of question 8.
8	The wall of proofs	Puzzle to prove statements.
9	The wall of challenges	Proposed problems (includes the 10 Questions) for the visitor to take a picture and take home. Postcards, or other material

Table 1. Section summary

- "Is there a point in the plane that is at the same distance to 3 given points?"
- "Do 4 lines in the plane always define 6 points of intersection?"
- "Do the diagonals of a parallelogram always cut in the middle points?"

The aim of a mathematician is to give answers to this kind of questions and show when a statement is always true or not. Some of the questions will refer to concepts seen in Steiner 10 Questions (introduction of the complete quadrilateral, to circumscribed circles, etc.). There will be no mention to a specific question.

- **Display 3: A quadrilateral on the floor**

  Text + panel with images + Image on the floor This section is build around a complete quadrilateral drawn on the floor Figure56. In this section there will be a complete quadrilateral drawn on the floor. Using some ropes, the aim is that the visitor will find the focal point in Steiner 10 Questions. The text and pictures on the walls will cover the construction of such a point

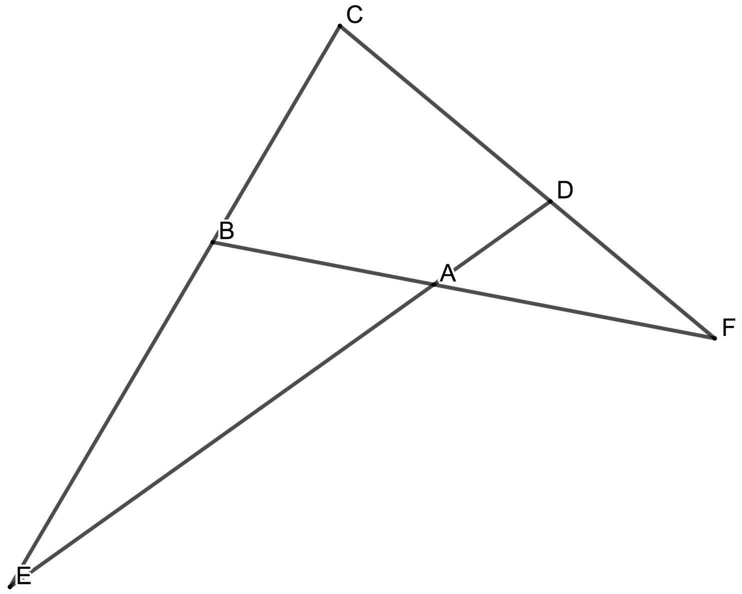


FIGURE 56. Complete quadrilateral

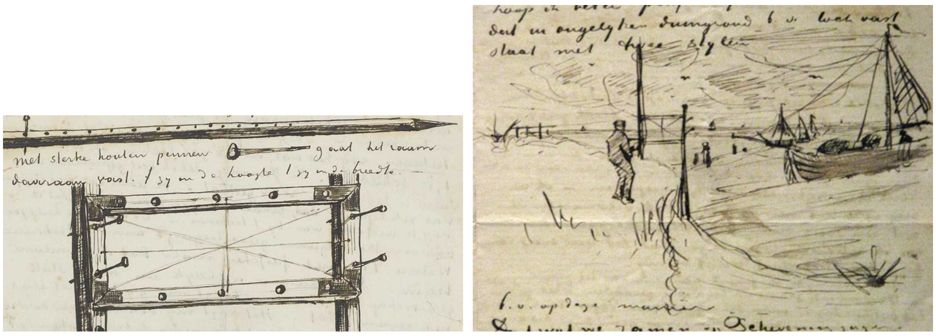


FIGURE 57. Vincent van Gogh description of the perspective frame

as well as Steiner's Questions 1 and 2. This can be verified using a set of sticks of the same length, a stick and a rope and a giant compass.

- **Display 4: The classroom**
  Blackboard + Smartboard + 4 school desks
  Four school desks are placed in front of a black board (with Steiner's Questions 3 and Steiner's Question 4 written in it) and a smart board. The visitor can paint on the board to try to figure out the answer to the questions. Are the points really aligned? On the smart board the visitor can use modern tools (like Geogebra) to learn about Steiner's Question 4.
- **Display 5: Mathematics or art?**
  Gadget + Text + Image In this display there is a description of the idea of perspective, and how you can find complete quadrilaterals in pictures. Vincent van Gogh descibed the perspective frame in his letters to his brother (Figure 57). A reproduction and instruction to use it will be displayed. Pictures of perspectives will be shown and the complete quadrilaterals will be displayed.

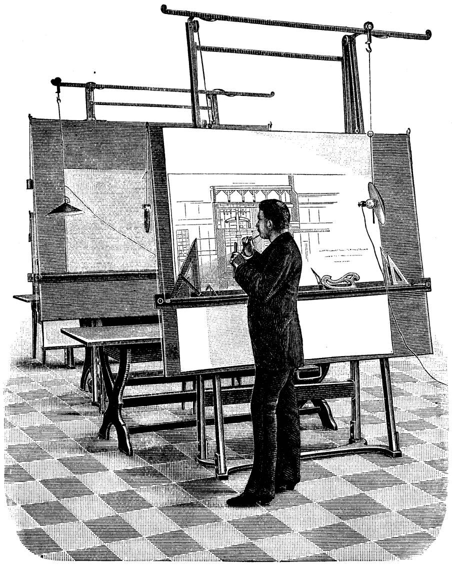


FIGURE 58. Architecture table

- **Display 6: Drawing in the past...**
  2 Table + Image One of the tables has drawing tools, a straight edge and a compass. It also has a picture of Steiner's Question 7. The visitor will be able to check using the tools that the lines are parallel. An architect "old" table (Figure 58) will be displayed next to it, with Steiner's Question 8 on it. This is going to be a handmade piece only black and white, similar to the ones used in the XIXth century.
- **Display 7: Drawing in the present...**
  Screen + text This screen will show the construction of Steiner's Question 9. The text will emphasize the idea of representation vs proof. Some mathematics are not exact, just an approximation. The construction takes 2 minutes to finish and there will be a steady image for 1 minute, and 1 minute of the quadrilateral moving.
- **Display 8: The wall of proofs**
  Metallic wall + Magnetic Pieces Sometimes we have to build our answers on top of other results! Euclid's postulates will be displayed on the top of this metallic wall. The rest of it will have space to build proofs. To this end, magnetic pieces with statements will be shaped in puzzle form. The idea is to start with a particular statement and build a formal proof [39].
- **Display 9: The wall of challenges**
  Images The last display will be a set of challenges without solution graded from 1 to 5 (being 5 the hardest ones). Among the proposed challenges the visitor will be able to distinguish Steiner 10 Questions as they will be framed in a different colour.

During all day long a guide characterized as a mathematician of the XIXth century will be around showing the exhibit around and giving explanations of the different displays, and stimulating visitors to take part in some of the displays.

4.2.3. *Economical Budget.* The estimated overall cost of this exhibit is 21 000€
Displays 1-3: 4000€
Displays 4-5: 4000€
Displays 6: 3000€
Displays 7-8: 4000€
Display 9: 3000 €

Audiovisuals: 3000€

4.2.4. *On the use of images.* The **Image copyright** will be taken into account. Each picture is in the public domain in its country of origin and other countries and areas where the copyright term is the author's life plus 70 years or less (which is the case of Spain and most European countries). Also there are going to be some pictures of own creation. Appendix A has some of the drawing of Steiner's 10 Questions.

4.3. **Proposal 2: Steiner's 10 Questions on the complete quadrilateral.**

4.3.1. *Concept.* When in 1827 Steiner proposed the 10 questions on the complete quadrilateral he was challenging other mathematicians to give their approach to the questions.

This exhibit wants to maintain the same idea. It is focused to give students and people that enjoy Mathematics the opportunity to face those same questions. With the help of images for each question, the results will be presented to the visitors for them to figure out a solution.

This exhibit is designed to be shown on a university hall, or a Mathematics Museum. As there wouldn't be any introduction to the topics (at least theoretical) its main public is people with some sort of background in Mathematics.

Since interaction with the visitor is important, two options will be implemented. A web-app that will act as a companion to the exhibit, and virtual space to send the answers to the problems. The web-app will give different approaches to each question, allow the user to move some of the points to see the evolution of each question depending on the quadrilateral, show how each question is constructed and a hint on where to start to solve the questions.

4.3.2. *Contents of the exhibit.* The exhibit will consist of pictures for every question, giving focus to some of the most difficult ones, and with almost no text. Each picture will be accompanied by the title (the statement of the question). Some auxiliary panels might be included to help a better understanding of the question and in particular Steiner's Question 8. As it is designed for a hall, it has been planned to need very little space. Figures 59, 60, 61, 62, and 63 show a possible layout using walls. A similar layout could be obtained using stand alone panels.

Prints are done on full coloured canvas with a frame 4 cm in pine wood.

There are 3 sizes, which gives an idea on the difficulty and complexity of each question.

The pictures will be accompanied by an audiovisual (either projected or with a floating screen), where the visitor will see the different pictures moving. The audiovisual goes over the 10 questions and the construction of each part. It takes 6 minutes to play the whole video.

Also an app will give more information to the visitors, how it is constructed, some important facts, and a hint on how to build a the proof.

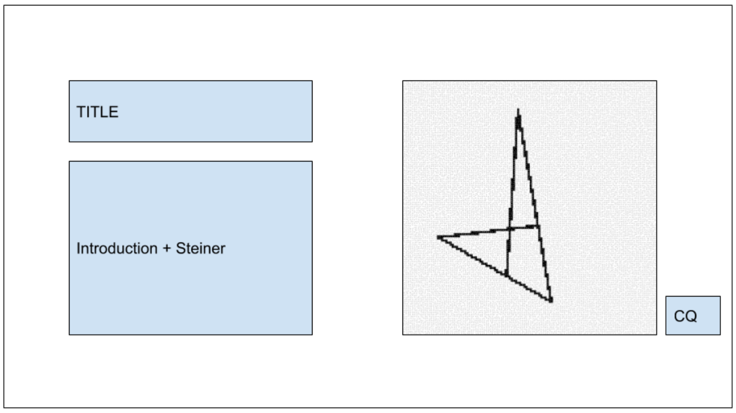


FIGURE 59. Layout of the introductory panel

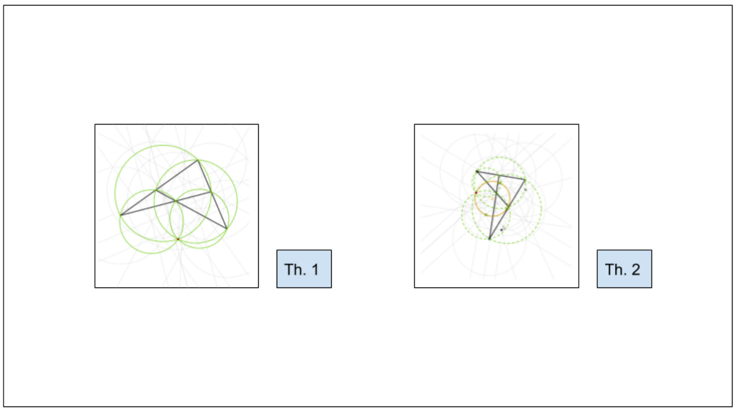


FIGURE 60. Layout of the Steiner's Questions 1 and 2

4.3.3. *Economical Budget.* The economical cost of creating this exhibit is around 2800€.

The main costs go for the prints to be produced:
3 Pictures 100cmx100cm 3x200€
4 Pictures 60cmx60cm 4x100€
8 Pictures 40cmx40cm 8x100€

Audiovisuals 1000€
Total 2800€

4.3.4. *On the use of images.* The **Image copyright** will be control: Each picture is in the public domain in its country of origin and other countries and areas where the

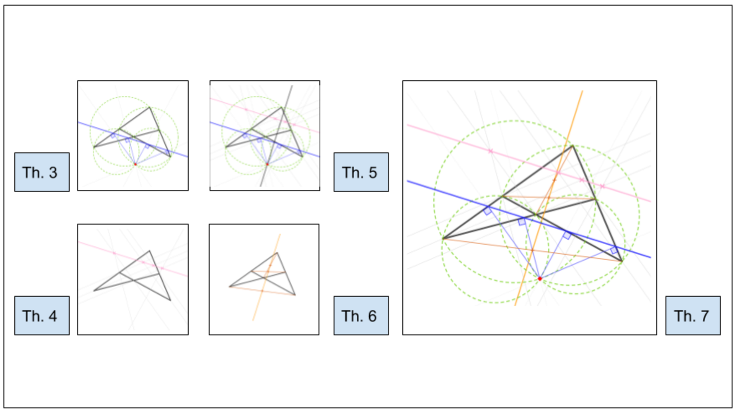


FIGURE 61. Layout of Steiner's Questions 3-7

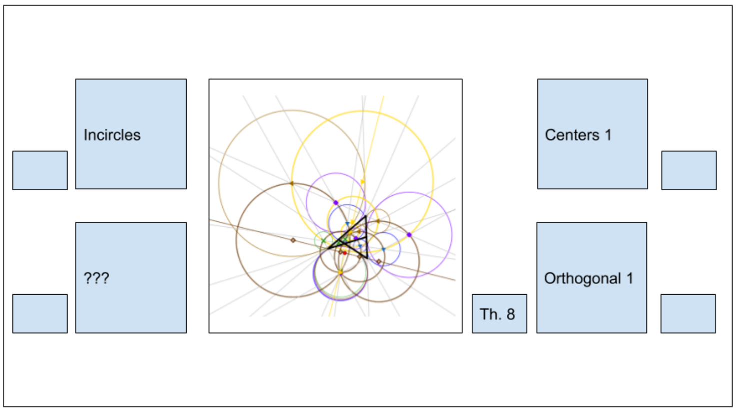


FIGURE 62. Layout of Steiner's Question 8

copyright term is the author's life plus 70 years or less. Also there are going to be some pictures of own creation. Appendix A has some of the images that would be exposed.

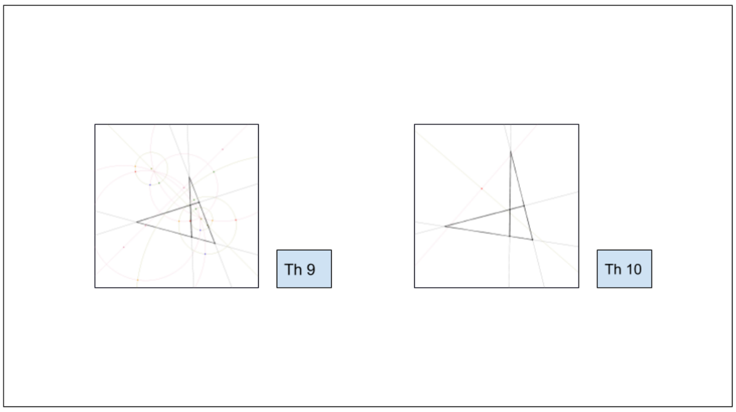


FIGURE 63. Layout of Steiner's Questions 9-10

Picture	Size
CQ	100cm x 100cm
Th.1	60cm x 60cm
Th.2	60cm x 60cm
Th.3	40cm x 40cm
Th.4	40cm x 40cm
Th.5	40cm x 40cm
Th.6	40cm x 40cm
Th.7	100cm x 100cm
Aux.1	40cm x 40cm
Aux.1	40cm x 40cm
Th.8	100cm x 100cm
Th.9	60cm x 60cm
Th.10	60cm x 60cm

TABLE 2. Size of the pictures by Theorem

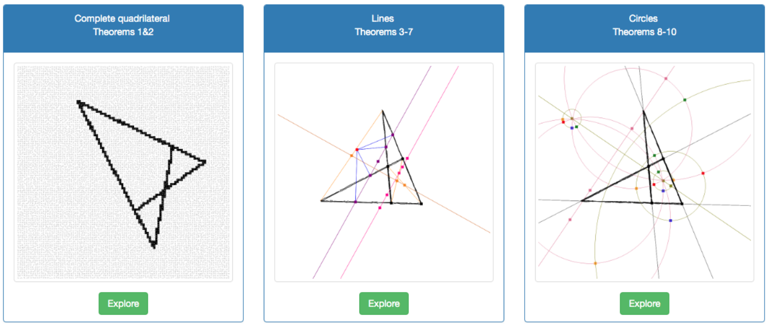


FIGURE 64. Front page

5. CONSTRUCTION OF THE PROGRAM

Dynamic Geometry has been around for more than 50 years. With the help of a computer, and with languages and programs like Logo (Turtle Geometry), Cabri, Cinderella and Geogebra, users can explore geometrical properties and make their own conjectures, or find counterexamples. Papert [40] already pointed out the advantage of using it as a tool for teaching and learning.

While their impact in the classroom has been studied, very few studies has been made on their impact in other areas like exhibits. Very few museums or exhibits use these tools in their displays. Although there are some parts that use computers and technology, they do not use dynamic geometry and they do not give the visitor the responsibility to make a conjecture.

One of the ideas of the project is to include some sort of audiovisual guide, or a place to explore Steiner's Questions.

The requirements for such an App are:

- Easy to execute
- No installation
- The content will be the theorems
- User should be able to move points to see how it evolves
- Automatic randomized movement
- You should be able to see the process of construction

5.1. **The working demo.** Three options have been considered: Geogebra [41], Cinderella [42], or own creation (with Javascript). Because the program is not difficult, and adapting someone else code to the specific needs is not always trivial, I have decided to do a web-app from scratch.

When trying to decide on the language to be used, we decided to use Javascript and HTML5. No installation will be required, since it is an interpreted language that can be executed with any browser. This direction is coherent with the direction that the other programs studied are taking. Both Geogebra and Cinderella were first created in JAVA, but now they have changed it to use Javascript instead.

The web-app starts with three sections, separated between Introductory Questions, Steiner's Questions 3-7 and 8-10. Figure 64 shows the options.

Once a question is selected, there is a submenu to decide what to do with that question. Four options have been created (Figure 65):

- Move. The first option allows the user to move one of the Black vertices that belong to the complete quadrilateral. The Figure adapts to the movement

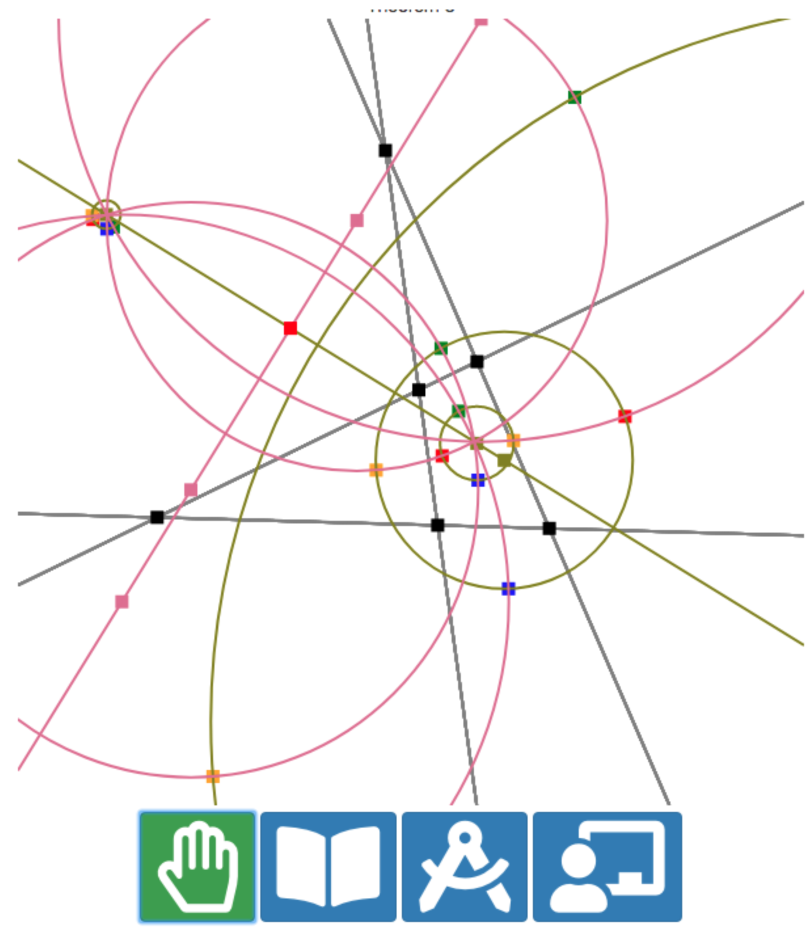


FIGURE 65. Bottom menu: Move, Static, Theory, Construction

- Static. Second option is the static version of the theorem, with the standard input
- Construction. Step by step process constructs each question
- Learning. A hint on how to prove the question is given in this last option

There is a working demo of the web-app that can be accesses at

www.geometrychallenge.com

5.2. **Points, lines and circles.** One of the most important choices to be made when creating a Dynamic Geometry Software is to decide on the data structure to store the main geometric elements. In the case of Steiner web-app, there are four type of elements:

- Points. Although it can be seen as a simple choice, it will influence the rest of the program. To be able to better mimic reality and Euclidean geometry, Cartesian coordinates were selected. Other options would have been homogeneous coordinates (the choice of Cinderella), vectors (geogebra), Polar coordinates, etc.
- Lines. The common choices are, storing a line as an equation, a vector plus a point, or as a function, a slope plus a point. The final choice was to use the last one. Javascript has the number *Infinite*, which allows us to be able to store vertical lines. The choice of storing a point and not just the y-intercept, is

because in vertical lines there is no y-intercept. The rest is easy to implement with this choice.

- Circles. We can store circles as a second degree equation, or as a center and a radius (or a center and a point). We have stored center + radius. To be able to work with intersection between circles and lines Cinderella uses complex coordinates. With them, an intersection with a circle has always 2 (not necessarily real) solutions. In our case if there is no real intersection, the result will be undefined.
- Complete Quadrilateral. To store this figure we will store the 6 intersecting points. Only 4 would be required, as the other 2 can be obtained from the first 4. Nonetheless to avoid repeated computations, the calculation is done only once, and the 6 values are stored.

5.3. **Difficulties encountered and how to overcome them.** As Kortenkamp states in his thesis [42] there are many difficulties when trying to work with dynamic elements. Some of these were found on the development of the program. Here are the main issues found:

- Continuity and coherence when moving elements. Figure 66 shows a representation of this problem. When point A moves passed a fixed point B, the intersections change order. This can be solved with the nearest element approach. Checking in each step that elements stay near their old selves. This solution works except for the next case.

FIGURE 66. Inversion of the intersection with the movement of A with respect to B

- Coming and going to infinity. If there is an asymptotic behaviour or the points go from a jump discontinuity we cannot use the previous solution. In our case this happens when the initial quadrilateral has parallel sides. This has been solved by not allowing the six points on the quadrilateral to exit the canvas.
- Degenerate cases. Figure 67 shows some of the cases. If some of the points are aligned when they shouldn't it generates weird figures that can break continuity of the problem. For this reason, there is a no zero area policy. Every triangle from the quadrilateral has to have area larger than some value.

There have been some other difficulties. These are a consequence of the choice of the language. Javascript is an interpreted language. One of the consequences is that the browser executes all the instructions and outputs the result. It is for this reason that Dynamic content is difficult to create. The solution is that a repeated event is going to be executed every 0.05 seconds. This event will be counting the number of times it has been executed, and then a partial execution will be made.

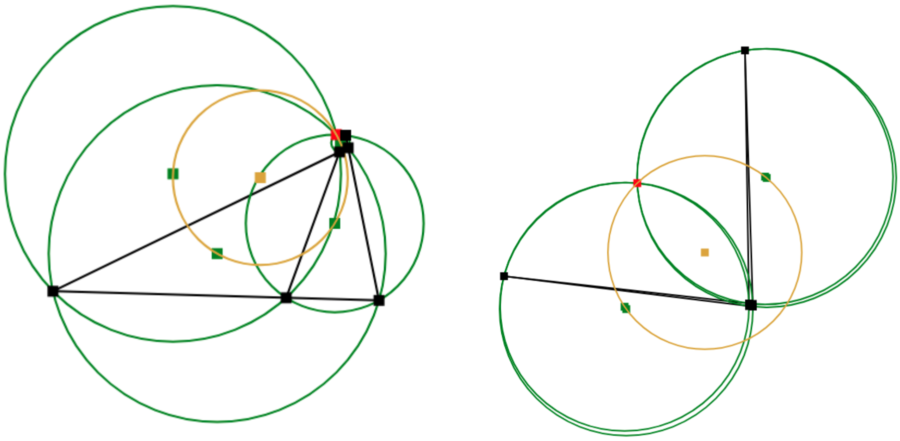


FIGURE 67. Two degenerate cases

UI functions		
Name	Input	Output
validSimplified	points	returns if the configuration is valid
updatePoints	updates the points after the movement	Array of points
mouseMoved	event	depending on whether it was pressed move the points
mousePressed	event	if pressed near a point select it or unselect if it had a selected point

TABLE 3. Some of the UI Functions

5.4. **The algorithms and the code.** As for the program on itself many things have been taken into account. We can say that there are three layers of coding. The first one in charge of the User Interface. Some of the functions and a short description can be found at Table 3.

In here we can discuss the validSimplified, which is a function that takes care of the difficulties mentioned before. Areas of triangles cannot be close to 0 and points should be inside canvas. Similarly updatePoints takes care of the continuous movement, and updates the points depending on which one you have selected. Finally the interaction of the mouse are covered by the last two functions.

A second layer has the coding of basic geometric functions, Table 4 shows a reference of some of the functions.

This functions are self-explanatory, and are the basis of the last part. More complex functions that are build upon these group. One of the paradigms that we have used is simulation. Many of the functions can be programmed doing some mathematics first and then just outputting the result, but because of the pedagogical approach, we have used the simulation of straightedge and compass. For example to get the coordinates of the orthocenter Figure 68 we can use a construction Algorithm 1, or we can just use a closed formula given the coordinates of the points. The first option is the one chosen.

Basic Geometry		
Name	Input	Output
drawPoint	point, colour	draws a point
drawLine	line, style	draws a line
drawSegment	point1, point2, style	draws the desired segment
evalLine	line, xcoordinate	y value of the line at that x
intersectLines	line1, line 2	returns intersecting point
makeLine	point1, point2	returns desired line
midPoint	point1, point2	returns the midpoint
makePerpendicularLine	point, line	perpendicular of line at point
pointsInLine	point1, point2, point3	returns true if aligned
circleIntersection	circle1, circle2	returns both intersections
intersectLineAndCircle	line, circle	returns 2 intersecting points (undefined if they don't intersect)
findDistance	point1, point2	distance between points (Pythagoras)
findDistancePointLine	point, line	distance between line and point

TABLE 4. Basic Geometric Functions

Algorithm 1: GetOrthocenter

Data: Triangle with vertices A,B,C
Result: Coordinates of the Orthocenter
begin
$SIDE_A \longleftarrow makeLine(B,C)$
$SIDE_B \longleftarrow makeLine(A,C)$
$HEIGHT_A \longleftarrow SIDE_A.getPerpendicularAt(A)$
$HEIGHT_B \longleftarrow SIDE_B.getPerpendicularAt(B)$
$ORTHOCENTER \longleftarrow HEIGHT_A.getIntersectionTo(HEIGHT_B)$
return $ORTHOCENTER$

For each of the more complex drawings 3 different functions were created:

$$get *** Triangle, \quad draw *** Triangle, \quad construct *** Triangle$$

These three functions are valid for circumcenter, orthoncenter, incenter, exocenter, and centroid, although this last one was not used in any of the Steiner's Questions. Finally there are functions for each Question (see Table 5).

We will take the circumcenter as an example. We can compare the algorithm to get the center, and the algorithm to construct it.

An intermediate result of the construction can be seen in Figure 69.

To find the circumcenter one could find the equation by computing the determinant

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0 \quad i = 1, \ldots, 3$$

where (x_i, y_i) are the coordinates of the vertex of the triangle.

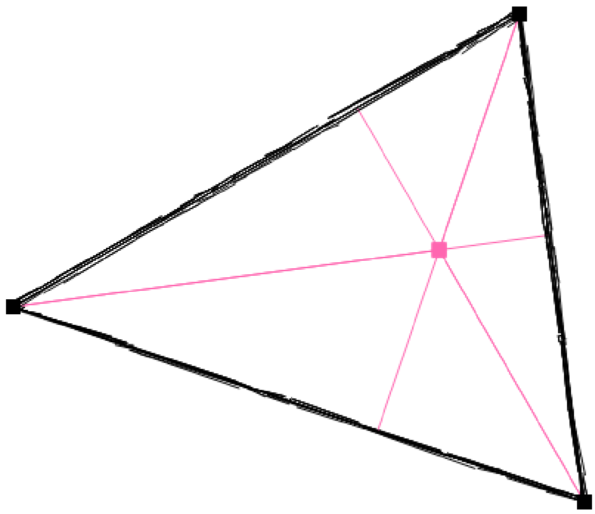


FIGURE 68. Orthocenters

Constructed Figures		
Name	Input	Output
getCircle	point1, point2, point3	circle through the points
makeAngleBisector	line1, line2, acute	returns the bisector of line (acute determines which one)
getOrthocenterTriangle	point1, point2, point3	get the coordinates of orthocenter
drawOrthocenterTriangle	point1, point2, point3, style	draws the orthocenter with the desired style
constructOrthocenterTriangle	validStep, point1, point2, point3, colorProcess, colorResult	constructs the orthocenter at validStep. While constructing it uses a colorProcess, after validStep it uses colorResult
getCircumcenter- Triangle	point1, point2, point3	gets the coordinates of the circumcenter
drawCircumcenterTriangle	point1, point2, point3, style	draws the circumcenter
constructCircumcenterTriangle	validStep, point1, point2, point3, colorProcess, colorResult	
question1	A,B,C,D,E,F	draws representation of Steiner's Question 1
question4	A,B,C,D,E,F	draws the line of orthocenters

TABLE 5. Geometrical construction figures

The algorithm used in the program to find it is more *construction* based. Algorithm 2 below shows this process. It is interesting to compare this with the *construction*

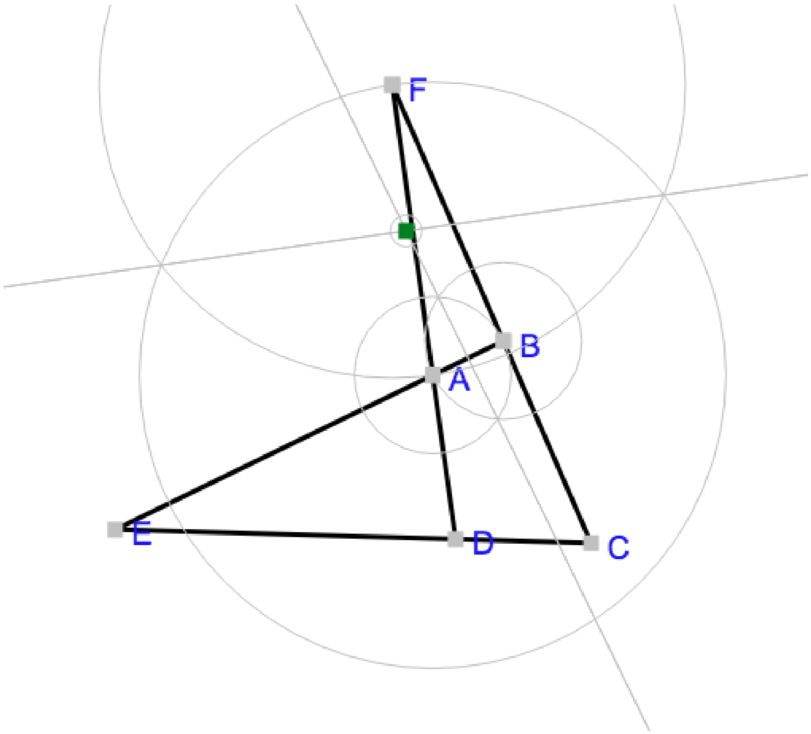


FIGURE 69. Step by step construction

Algorithm 2: getCircumcenterTriangle

Data: Triangle with vertices A,B,C
Result: Coordinates of the Circumcenter
begin
$SIDE_A \longleftarrow makeLine(B, C)$
$SIDE_B \longleftarrow makeLine(A, C)$
$BISECT_A \longleftarrow SIDE_A.getPerpendicularAt(MIDPOINT(B, C))$
$BISECT_B \longleftarrow SIDE_B.getPerpendicularAt(MIDPOINT(A, C))$
$CIRCUMCENTER \longleftarrow BISECT_A.getIntersectionTo(BISECT_B)$
return $CIRCUMCENTER$

---

step by step of the circumcenter. In this case we want to see an image similar to Figure 69. This is obtained with Algorithm 3. In this case the variable $STEP$ allows us to control the flow of the construction. Every construction function has a similar aspect, in this way we can subtract the *time* we are using and if it ends up being negative it will not draw it (instead of doing an IF at each moment). Also it is important to notice that the construction takes a certain amount of *time*. For example in Algorithm 3 we can see that *constructCircle* takes 2 units of time. The overall construction takes 12 units of time. At the end variable $STEPS$ is translated to seconds. Because Javascript is executed and then after prints the result, we have to set up a timer for which the movement will be produced. This is done with the setInterval

$$Interval = setInterval(function()\{moveTheorem1();\}, 50);$$

We can see that every 0.050s the function will be called. Inside $moveTheorem1()$ we have a $step = step + 0.1$; this means that $1Step = \frac{0.05}{0.1}s = 0.5s$. So constructing one circumcircle takes $6s$. Some of the questions have a lot of steps to be build. In these cases the algorithm has been adapted to construct part of the figures at the same time. This can be changed if it is considered to be confusing.

Algorithm 3: ConstructCircumcenterTriangle

Data: Triangle with vertices A,B,C, STEP
Result: circumcenter. Constructs circumcenter.
begin
  if *Step >= 0* **then**
    $SIDE_A \longleftarrow makeLine(B, C)$
    $SIDE_B \longleftarrow makeLine(A, C)$
    $R_{BC} \longleftarrow getDistance(B, C)$
    $C_{BC} \longleftarrow constructCircle(B, R_{BC}, STEP)$
    $C_{CB} \longleftarrow constructCircle(C, R_{BC}, STEP - 2)$
    $P_1, P_2 \longleftarrow getIntersectionCC(C_{BC}, C_{CB})$;
     $BISECT_{BC} \longleftarrow constructLine(P_1, P_2, STEP - 4)$
    $R_{AC} \longleftarrow getDistance(A, C)$
    $C_{AC} \longleftarrow constructCircle(A, R_{AC}, STEP - 6)$
    $C_{CA} \longleftarrow constructCircle(C, R_{AC}, STEP - 8)$
    $P_1, P_2 \longleftarrow getIntersectionCC(C_{AC}, C_{CA})$;
     $BISECT_{AC} \longleftarrow constructLine(P_1, P_2, STEP - 10)$
    $CIRCUMCENTER \longleftarrow BISECT_{BC}.getIntersectionTo(BISECT_{AC})$
  return $CIRCUMCENTER$

The final examples are for some of the Steiner's 10 questions. In particular Steiner's Questions 1 and 4. We can see the algorithms in Algorithm 4 and 5. Notice that on Steiner's Question 1 if the intersection can change we need to compare the result obtained with A (that belongs to the circles). This is important since it was one of the difficulties stated.

The second example of algorithm is to find the orthocentric line (Steiner's Question 4). This can be obtained by drawing each orthocenter. In here triangles ABF and triangle DCF cannot have the same orthocenter, but triangles ABF and BDE, or DCF and BCE could have the same orthocenter. We can see that and we are already taking it into account when creating the program.

Although there are more options that could be added, the aim of this part was to create a working demo, and this is why it was the focus of the project.

Algorithm 4: Steiner's Question 1

Data: Quadrilateral with vertices A, B, C, D, E, F
Result: Coordinates of P (Result of Steiner's Question 1)

```
begin
    T_ABF ⟵ makeTriangle(A, B, F)
    C_ABF ⟵ makeCircumcircle(T_ABF)
    T_ADE ⟵ makeTriangle(A, D, E)
    C_ADE ⟵ makeCircumcircle(T_ADE)
    I ⟵ makeIntersections(C_ABF, C_ADE)
    //I contains two points
    if I[0] equals A then
        RESULT ⟵ I[1]
    else
        RESULT ⟵ I[0]
    return RESULT
```

Algorithm 5: Steiner's Question 4

Data: Complete Quadrilateral A, B, C, D, E, F
Result: R' Orthocentric Line

```
begin
    cleanCanvas()
    P ⟵ getMiquelPoint(A, B, C, D, E, F)
    𝒯 ⟵ trianglesABF, DCF, BCE, BDE
    forall t ∈ 𝒯 do
        H[i] ⟵ getOrthocenter(t)
        drawPoint(H[i])
    HLine ⟵ makeLine(H[0], H[1])
    return HLine
```

6. Conclusions

The main objective of the paper was to build a proposal for an exhibit of Steiner's 10 theorems. These theorems are a good choice of topic for an exhibit because they are not well known, they are approachable (the mathematics behind can be simple) and they are interesting and beautiful. The creation of the proposal has been achieved, and two different versions have been given depending on the visitors, and on the vision. A more ambitious exhibit was proposed, where a whole room was covered with gadgets, activities and technology. A plain approach was given, with much lesser cost, and maintaining Steiner's approach to the 10 Questions as challenging problems to the viewer.

In the first part we have seen that current Science museums *do not* show math as a rigorous discipline, and *do not* discuss the abstract though, or talk about proofs. It just shows mathematics as a utilitarian tool. The lack of rigor sometimes makes the museums state false (or incomplete information). Most Science museum are focused for young children. On the other hand we have visited some Mathematical Museums (MOMATH and MMACA). These have a broader approach and vision. They do focus also on adults, mainly through conferences, or workshops. Their exhibits are more rigorous, but do not work around geometry. The abstraction is not found, the visitor needs the objects to engage, and depending on the museum there is more or less technology.

On the mathematical aspect of Steiner's 10 questions we have seen that a lot of them could be solved by angle chasing, which, in my opinion, shows the beauty of geometry. We have seen how mathematics are constructed on top of other mathematician's ideas, and how proposing challenges to other people is not necessary an easy task. Although a simple solution for Question 10 was not found, an outline of a more complicated proof was given and solutions for the other nine Questions were given. With the help of computer imaging, it is easier to create the figures, to help with the study, making conjectures and giving proofs.

Finally a program and an exploration on Dynamic Geometry was done. A working demo has been produced for the reader to try. Although the algorithmic requirements were not difficult, the overall experience to create the program has been very positive. The difficulties found in the creation of a more complete software called Cinderella were also experimented during the development of the webApp and solved satisfactorily.

6.1. **Further work.** There are many aspects for future work. For example with more time and resources, the first part, the study of current museums, should be expanded to include Asian museums. The cultural differences would enrich considerably the section. I would also like to develop an index (similar to what Wagensberg presented in [3]) with specific parameters to study the museums and their impact in a more rigorous way. Finally there is another event that would be fundamental to attend, it is the Bridges Math Art Conferences [43], which are held every year around the month of July, and relate Mathematics, Architecture, Art, Music, etc. They bring together Mathematicians and people from other disciplines to build an exchange atmosphere and share ideas like this exhibit.

For the second part, the mathematical approach to the theorems, there are many places to improve, or investigate. First of all one could try to solve the same questions using other mathematical tools. Projective geometry, complex numbers, inversions, or barycentric coordinates are some techniques and ideas that could be tried. Steiner's Question 10 can be studied more in detail, and could be used to show different (and more complex) techniques. Finally there are some constructions of complete quadrilaterals using the incenters in Steiner's Question 8 that I would like to try. Investigating other theorems on the Complete Quadrilateral, or checking the importance of particular cases (like the quadrilateral being inscriptible) in the proofs would also be nice.

Being able to make the exhibit become a reality (and not only a proposal) would be the further work that could be done in this third part. Also there could be a publication on local mathematical journals for teacher/students with the reedition of the theorems. From the final section, we could use homogeneous coordinates to make the program more robust, more options could be implemented (zoom, adding elements, etc).

6.2. **A final though.** As Magritte meant with his painting *Ceci n'est pas une pipe* (Figure 70) [44]. Any object, painting or idea in a museum loses it's authenticity, and its purpose changes to become something it is not, a tool to teach and inspire younger (or not) generations.

FIGURE 70. The treachery of Images. Magritte 1929

But mathematics, and geometry, are different. A triangle is not something fixed, it's an abstract figure with special properties. And the complete quadrilateral holds many secrets, Steiner proposed 10 questions to be solved, and it challenged mathematicians to find the answer, and they did give answers, but the work is not necessarily done. Maybe Steiner had a simpler answer in mind, one can never know but can just guess that he saw something we haven't.

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Appendix A. Steiner's 10 Questions

Here you can find Steiner's 10 Questions with their images. This is designed as support material for the proposals.

Four lines **A**, **B**, **C**, **D**, intersecting two by two in six points and, in consequence belong to a same plane.

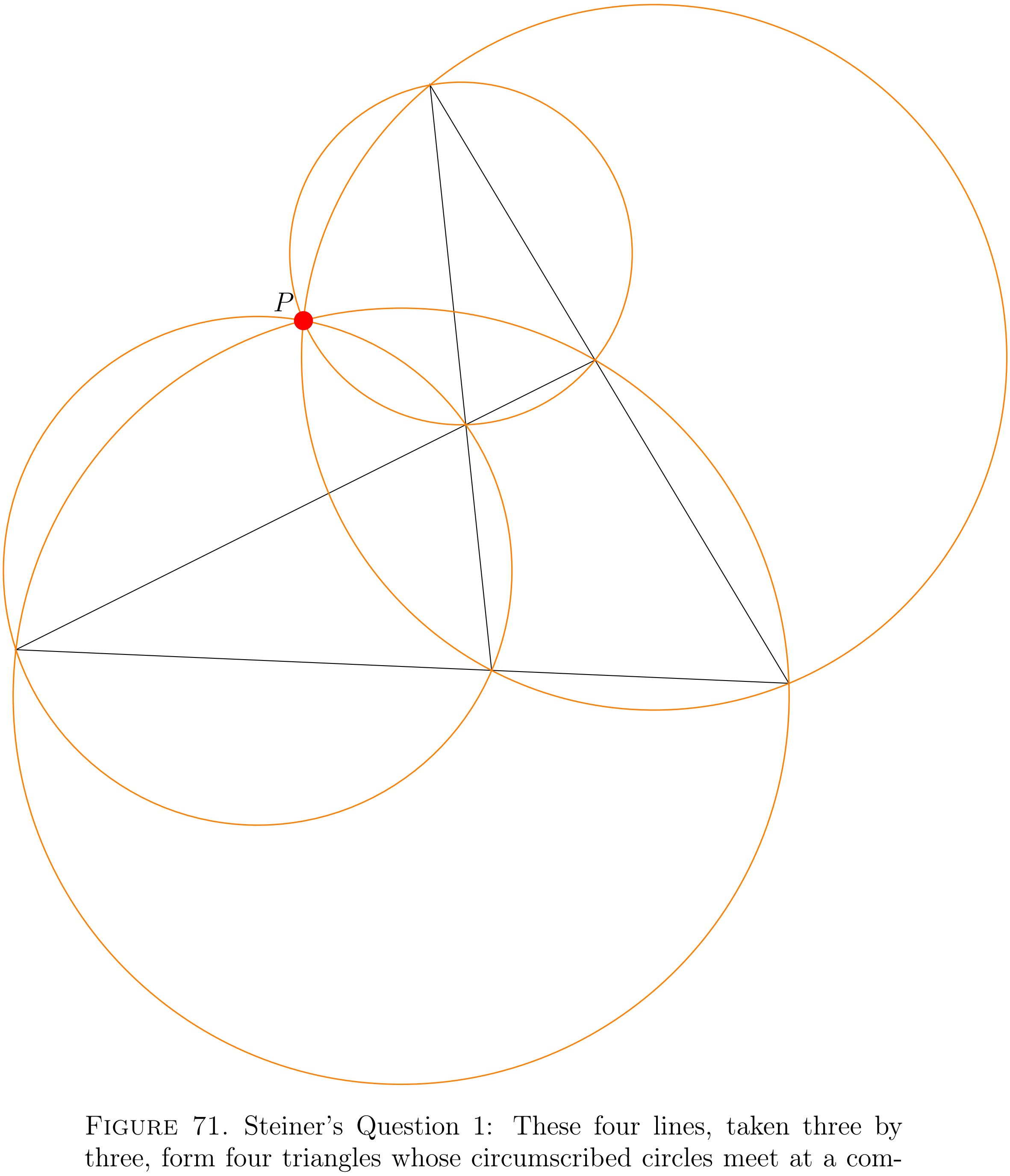

Figure 71. Steiner's Question 1: These four lines, taken three by three, form four triangles whose circumscribed circles meet at a common point **P**.

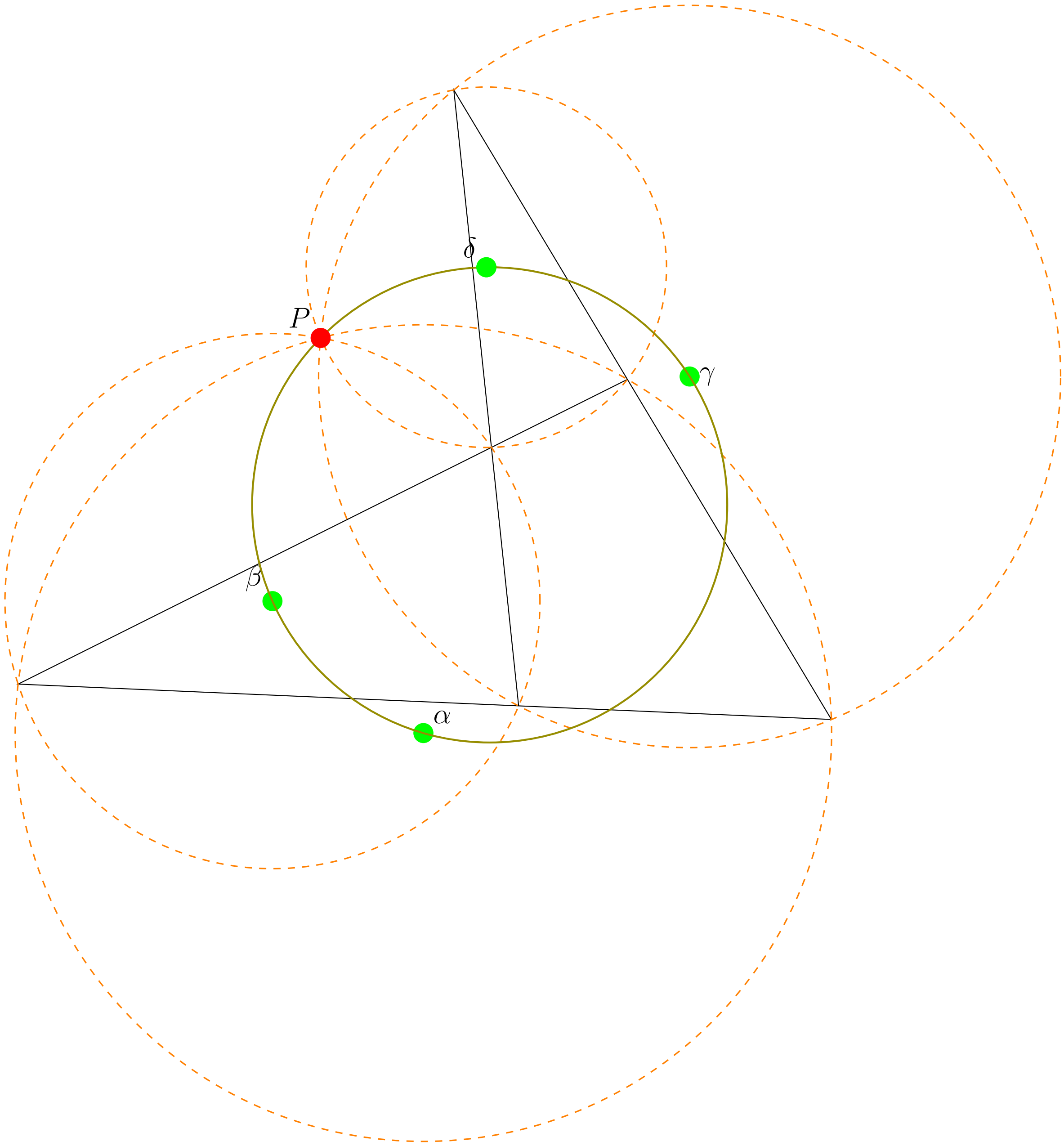

FIGURE 72. Steiner's Question 2: The centers α, β, γ, δ with point**P** lie on a fifth circle.

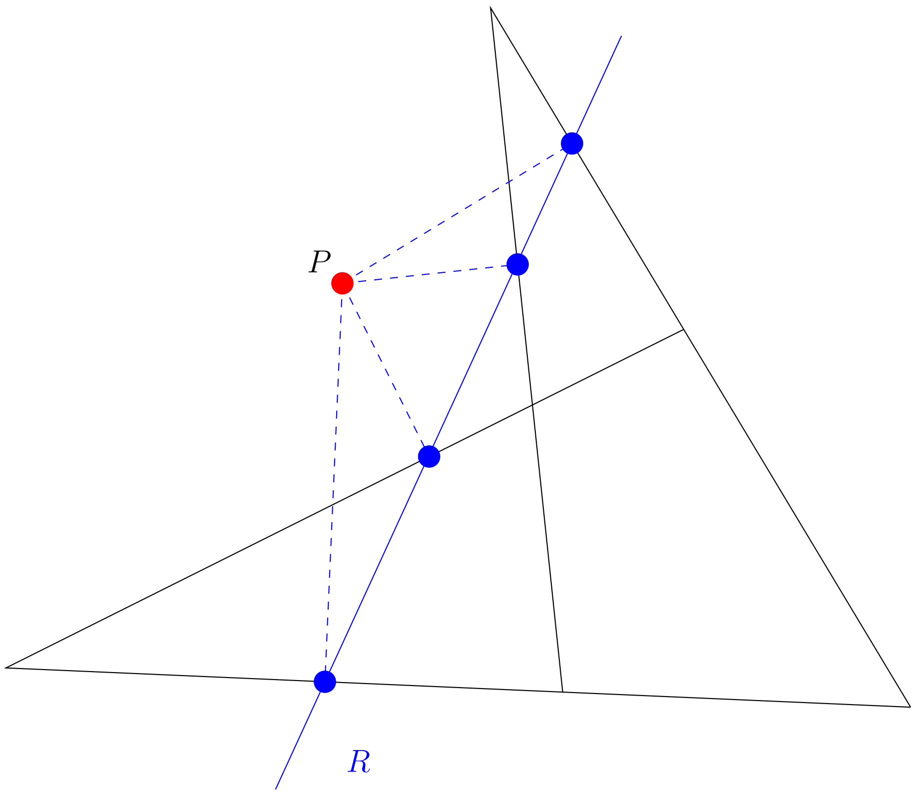

FIGURE 73. Steiner's Question 3: The feet of the perpendiculars to the directions **A**, **B**, **C**, **D** from **P** belong all four to the same line **R**, this property is exclusive for point **P**.

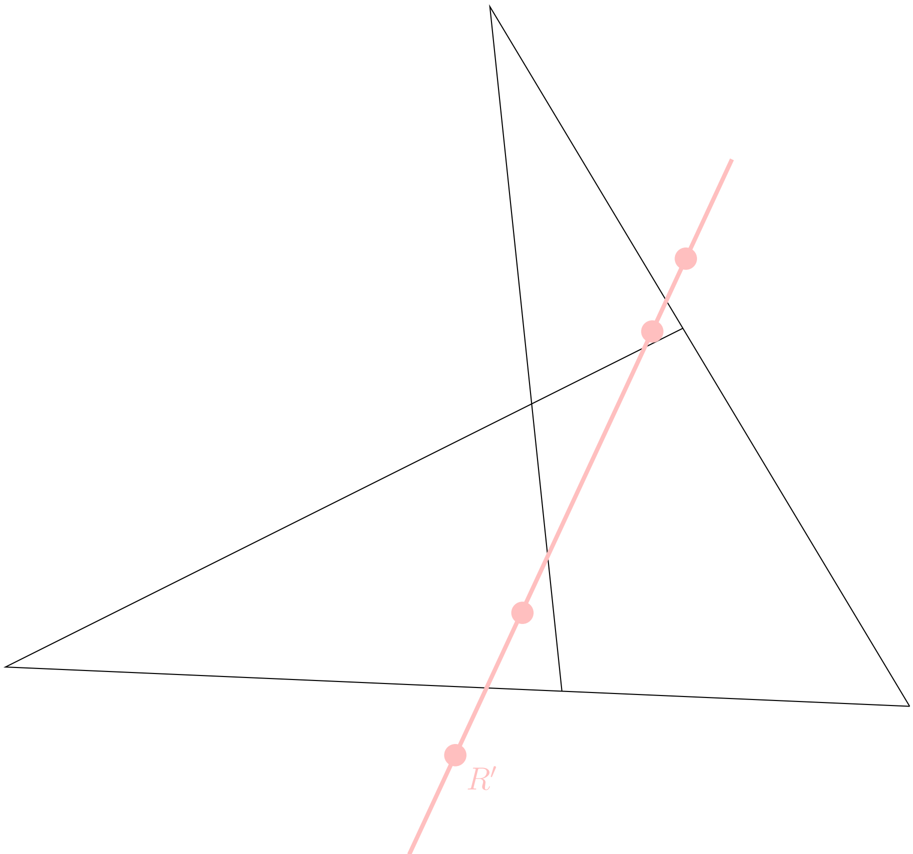

FIGURE 74. Steiner's Question 4: The meeting points of the perpendiculars from the vertices to the opposing sides of the four triangles (1) belong to the same line **R'**.

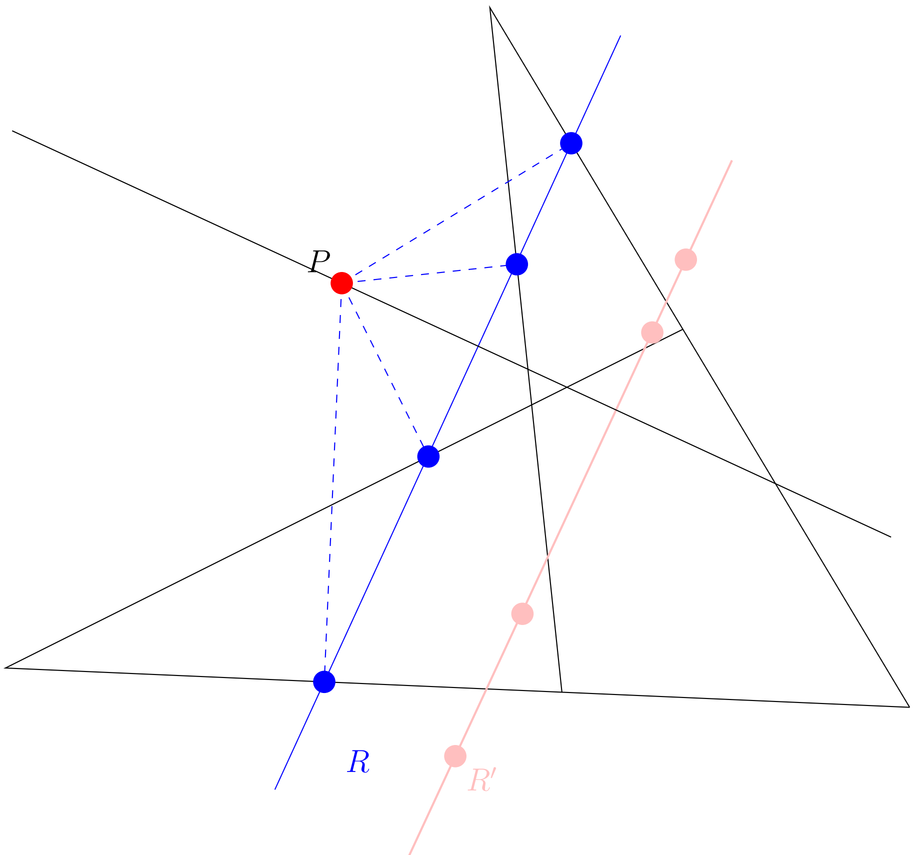

FIGURE 75. Steiner's Question 5: Lines **R** and **R'** are parallel, and line **R** goes through the middle of the perpendicular from **P** to **R'**.

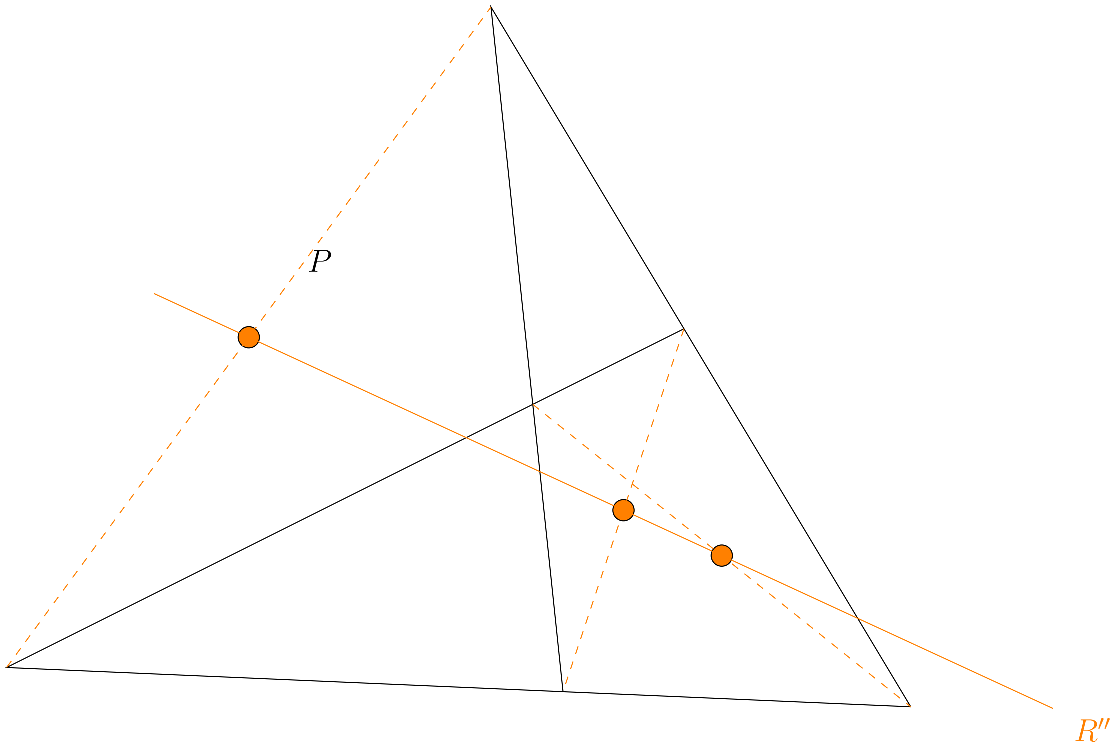

FIGURE 76. Steiner's Question 6: The midpoint of the diagonals of the complete quadrilateral created by the four lines **A**, **B**, **C**, **D**, belong all 3 to the same line **R"** (*Newton line*).

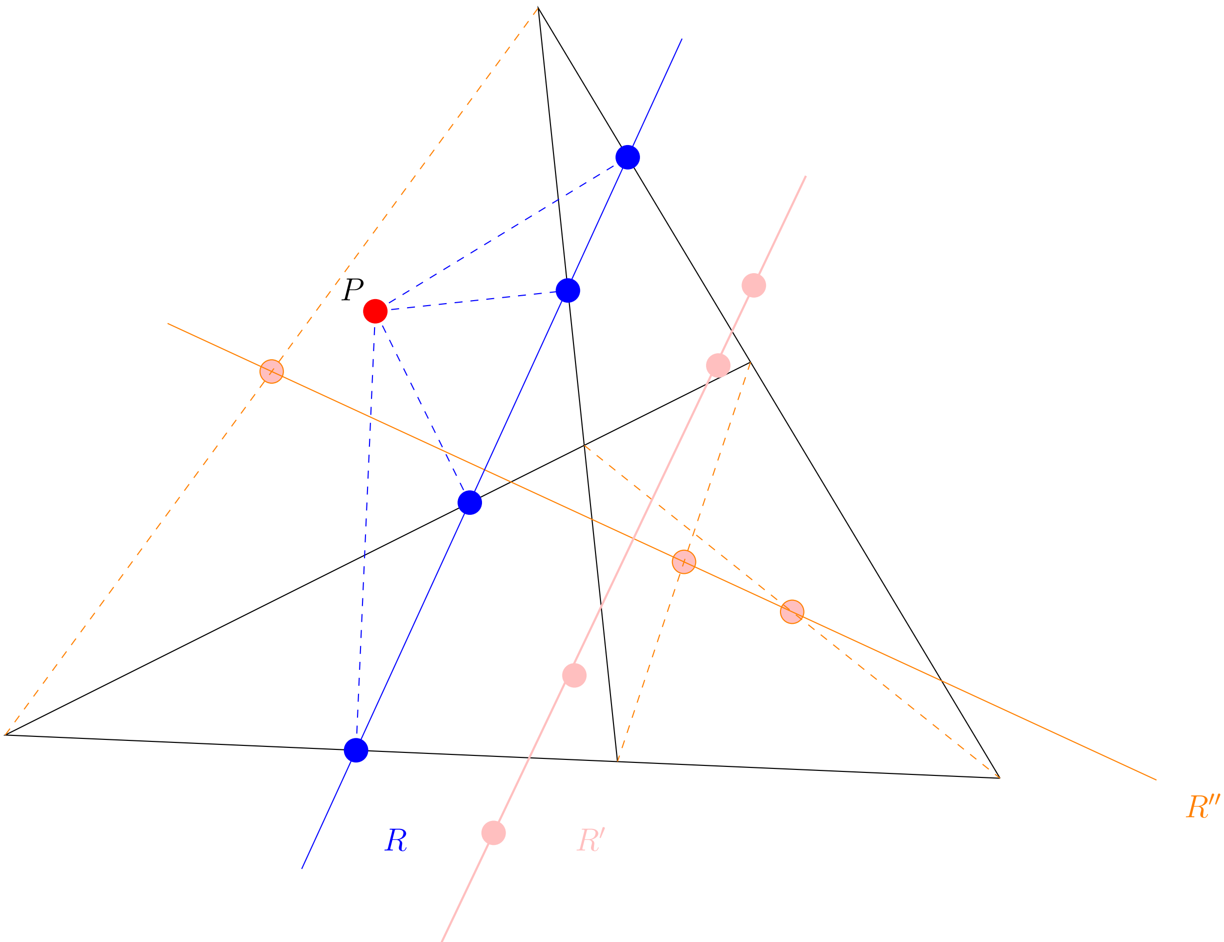

FIGURE 77. Steiner's Question 7: Line **R"** is a common perpendicular to both lines **R** and **R'**.

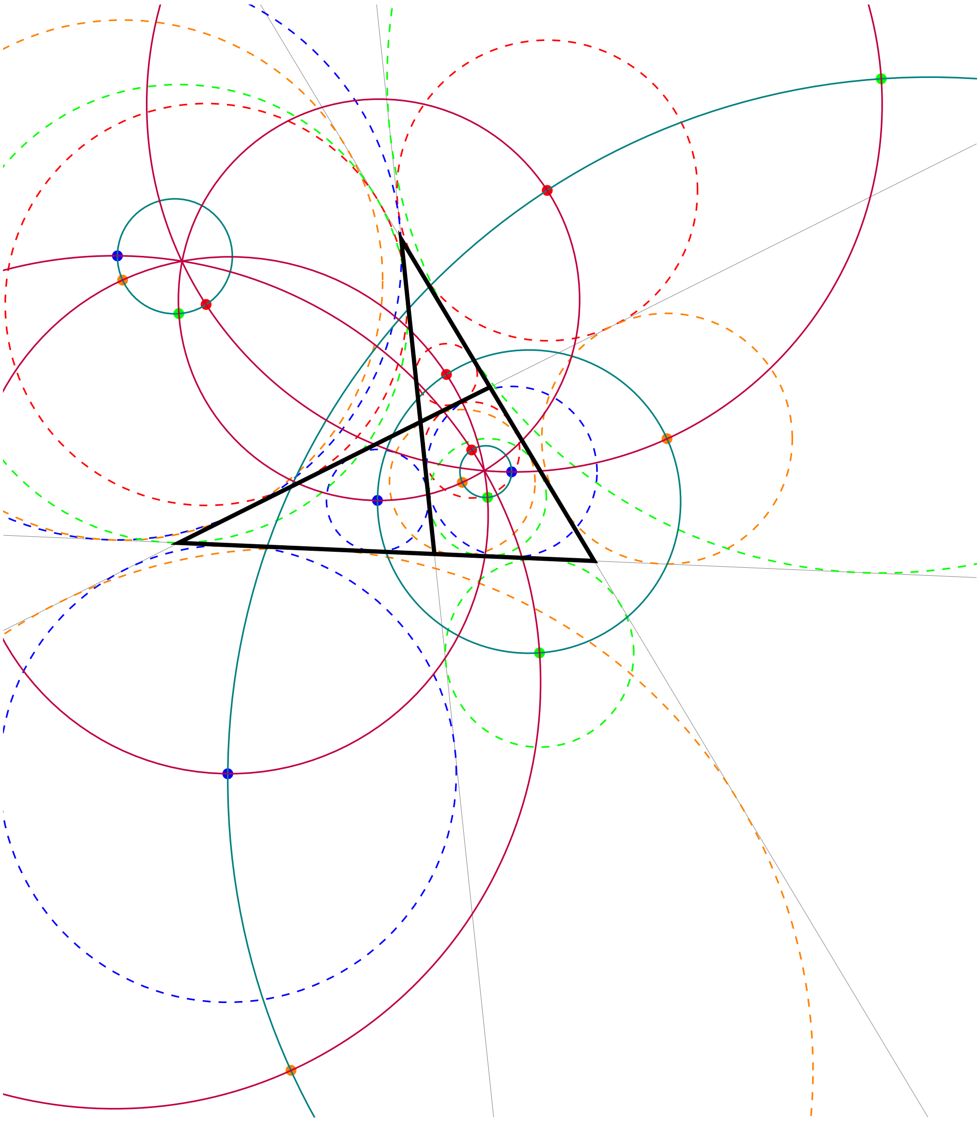

FIGURE 78. Steiner's Question 8: For each of the four triangles (1) there is an inscribed circle and three excircles, which makes in total *sixteen* circles; the centers of which are four by four in the same circle, creating *eight* new circles.

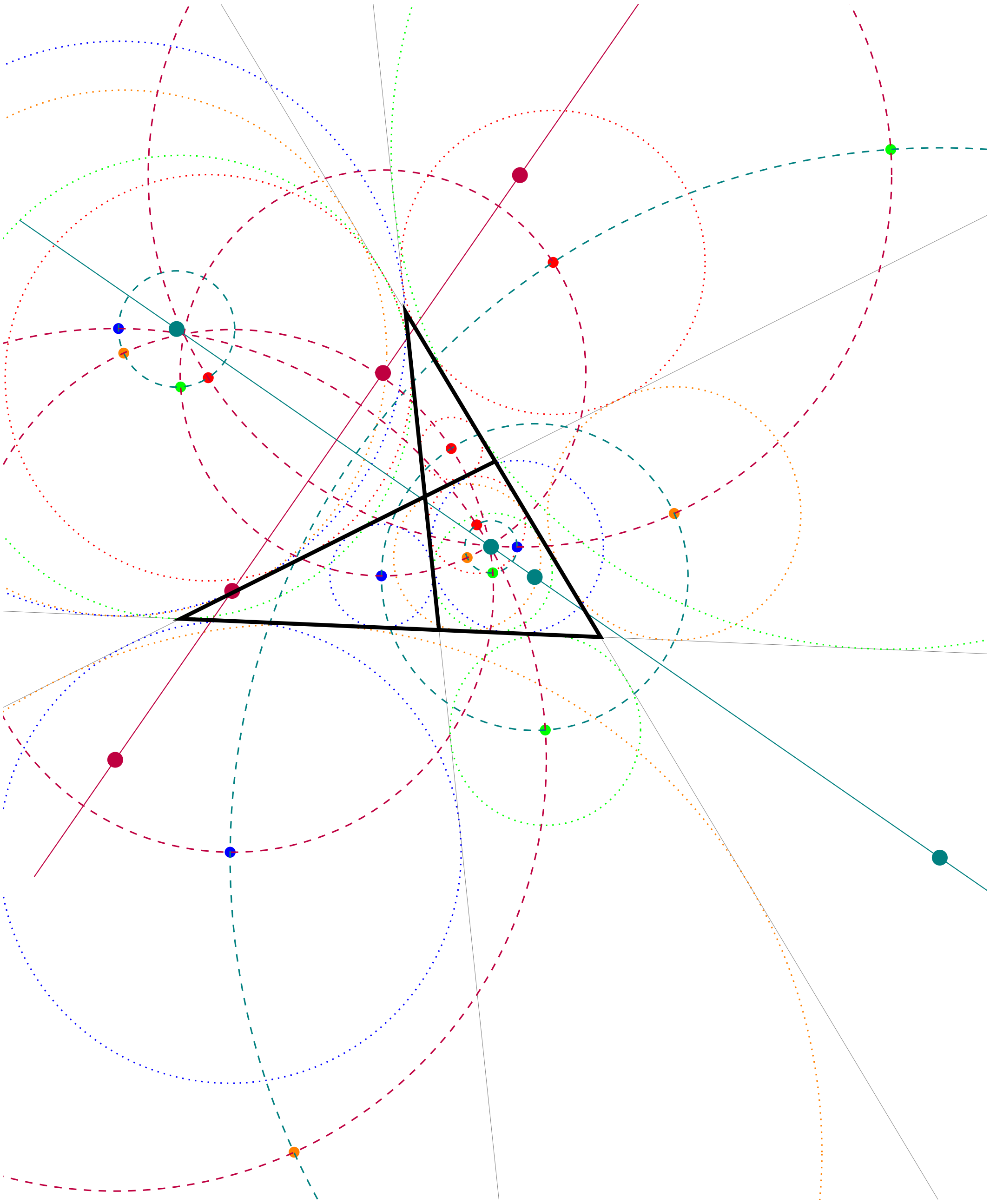

FIGURE 79. Steiner's Question 9: These new eight circles can be divided in two groups such that each of the four circles in one these group intersects orthogonally all the circles of the other group; we can conclude that the centers of the circles of both groups belong to two lines one perpendicular to the other.

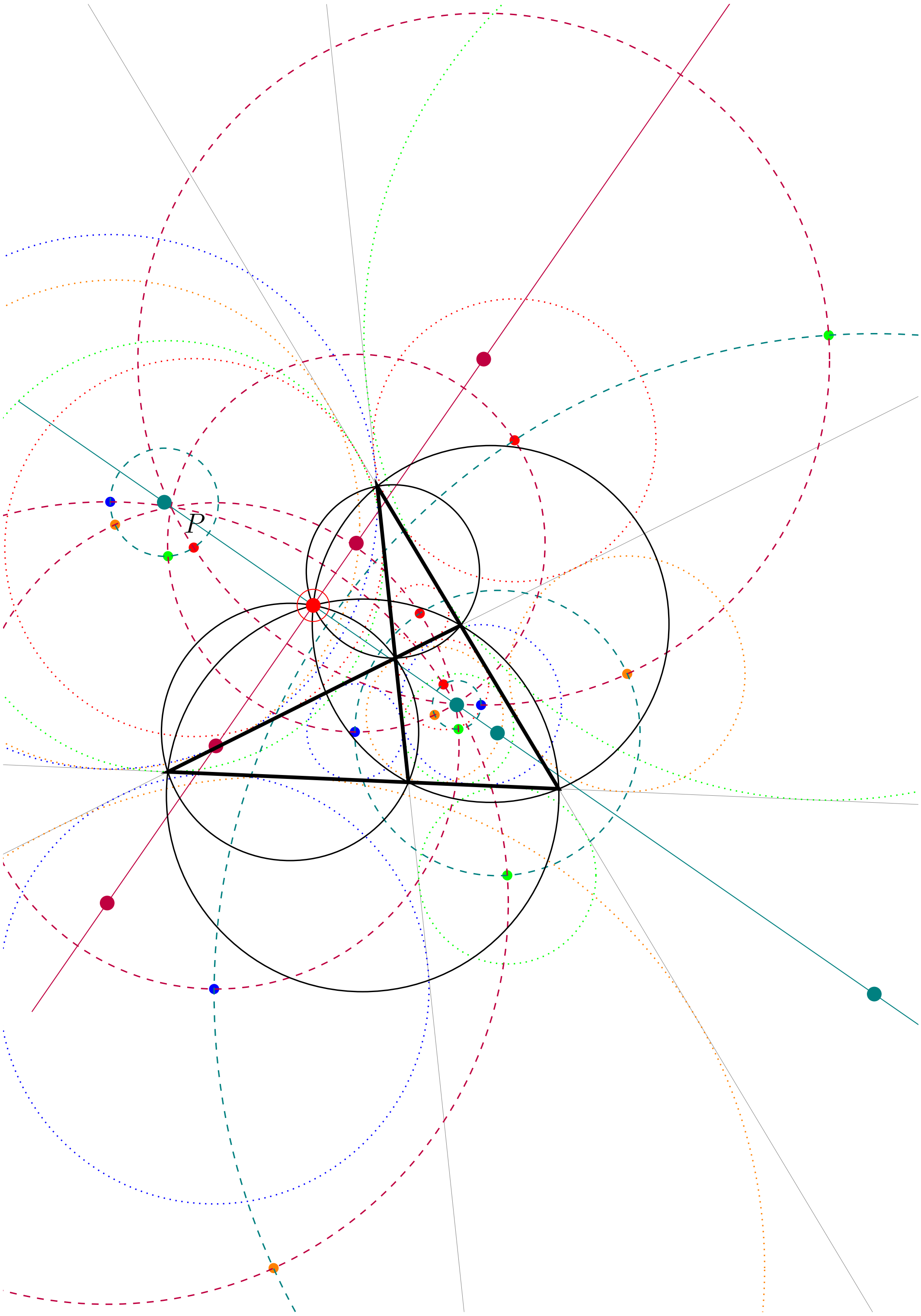

FIGURE 80. Steiner's Question 10: Finally these last two lines meet at point **P**, mentioned previously.